Lecture 5

Wednesday, January 15, 2020

Consider the set $V = \left\{ \begin{bmatrix} a & b \\ i(a+c) & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$.

One can see that V is a subset of $M_{2\times 2}(\mathbb{C})$.

V is not a subspace of $M_{2\times 2}(\mathbb{C})$ when viewed as a vector space over $\mathbb{C}$. In other words, V is not a vector space over $\mathbb{C}$. The reason is that V is not closed under scaling by a complex number. For example, the matrix

$$A = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix}$$

belongs to V (with $a = 1$, $b = c = 0$), but the matrix

$$iA = \begin{bmatrix} i & 0 \\ -1 & 0 \end{bmatrix}$$

doesn't belong to V.

Nevertheless, V is a vector space over $\mathbb{R}$. How so? We have a general rule as follows.

* Theorem.

If V is a vector space over $\mathbb{C}$ then it is also a vector space over $\mathbb{R}$ and over $\mathbb{Q}$. If V is a vector space over $\mathbb{R}$ then it is also a vector space over $\mathbb{Q}$. In short, if V is a vector space over a big field, then it is a vector space over smaller fields.

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

We know that $M_{2\times 2}(\mathbb{C})$ is a vector space over $\mathbb{C}$. Therefore, it is also a vector space over $\mathbb{R}$. One can show that V is a subspace of $M_{2\times 2}(\mathbb{C})$ (when viewed as a vector space over $\mathbb{R}$) by checking that

(1) $0 \in V$,

(2) V is closed under addition,

(3) V is closed under scaling (by real numbers).

What is a basis and the dimension of V?

We can rewrite V as

$$V = \left\{ a\begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}}_{A_3} \right\}$$

Put $B = \{A_1, A_2, A_3\}$. We see that $V = \operatorname{span} B$. To say that B is a basis of V, one needs to check if B is linearly independent.

Consider the equation $c_1 A_1 + c_2 A_2 + c_3 A_3 = 0$ with unknowns $c_1, c_2, c_3 \in \mathbb{R}$. This equation is equivalent to

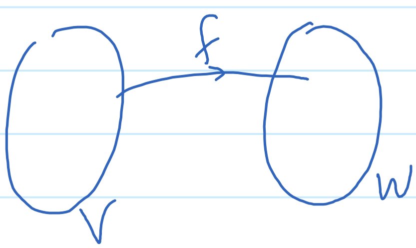
$$\begin{bmatrix} c_1 & c_2 \\ i(c_1+c_3) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, $c_1 = c_2 = c_3 = 0$. We conclude that B is a basis of V. Moreover,

$$\dim_{\mathbb{R}} V = 3.$$

* Linear maps.

Let V and W be vector spaces over F. There are many maps from V to W. Think of the case $V = W = \mathbb{R}$. There are a lot of maps from $\mathbb{R}$ to $\mathbb{R}$. We will be considering a useful type of maps called linear maps.



A map $f : V \to W$ is said to be linear (over F) if it satisfies two following properties:

(1) Additive:

$$f(v+w) = f(v) + f(w) \quad \forall v, w \in V$$

("First add, then apply f" is equal to "first apply f, then add")

(2) Scalar multiplicative:

$$f(cv) = cf(v) \qquad \forall c \in F,\ v \in V.$$

("First scale, then apply f" is the same as "first apply f, then scale".)

* Most maps are not linear, for example $f(x) = x^2$ (violate additive rule), $f(x) = \sin x$ (violate both additive and scaling rule), ...
  The set of all linear maps from V to W is denoted as $\mathcal{L}(V,W)$.

* A useful fact about linear maps is that they always map to zero vector to the zero vector. To see why, one can apply the additive rule for $v = w = 0$:

$$f(0+0) = f(0) + f(0)$$

This implies $f(0) = f(0) + f(0)$. By the Cancellation law (Homework 1), we obtain $f(0) = 0$.

Ex: Let V be the set of all smooth functions from $(0,1)$ to $\mathbb{R}$. By smooth, we mean infinitely differentiable. One can check without difficulty that V is a vector space over $\mathbb{R}$.

The differential operator $D: V \to V$, $D(u) = u'$ is a linear map. This is because

$$(u+v)' = u' + v'$$

$$(cu)' = cu'.$$

Ex: Let V be the space of all continuous functions from $[0,1]$ to $\mathbb{R}$. The integral operator $I: V \to \mathbb{R}$, $I(u) = \int_0^1 u(x)\,dx$

is a linear map. This is because

$$\int_0^1 (u(x) + v(x))\,dx = \int_0^1 u(x)\,dx + \int_0^1 v(x)\,dx,$$

$$\int_0^1 cu(x)\,dx = c\int_0^1 u(x)\,dx$$

Ex: The determinant map $\det : M_{2\times 2}(\mathbb{C}) \to \mathbb{C}$,

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

is not linear because it violates the addition rule:

$$\left.\begin{aligned} \det\left(2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) &= \det\left(\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}\right) = -8 \\ 2\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) &= 2(4-6) = -4 \end{aligned}\right\} \text{different}$$

Ex: Problem 2 on the worksheet.

$f : \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$ Recall: if $z = a + bi$ then $\bar{z} = a - bi$.

(a) Show that f is a linear map over $\mathbb{R}$.

We need to check two properties:

- Check if f is additive:

  That is to check $f(z_1 + z_2) = f(z_1) + f(z_2)$ $\forall v, w \in \mathbb{C}$

  Let $z_1, z_2 \in \mathbb{C}$ We want to show

  $$f(z_1 + z_2) = f(z_1) + f(z_2).$$

  That is to show

  $$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2.$$

  This is a well-known property of complex numbers.

- Check if f is scalar multiplicative (over $\mathbb{R}$):

  That is to check $f(cz) = cf(z)$ $\forall c \in \mathbb{R}, z \in \mathbb{C}$.

  Let $c \in \mathbb{R}$, $z \in \mathbb{C}$. We want to show

  $$f(cz) = cf(z).$$

  That is to show

  $$\overline{cz} = c\bar{z}.$$

This is a well-known property of complex numbers.

(b) f is not a linear map over $\mathbb{C}$ because it violates the scalar multiplication rule. For example,

$$\left.\begin{aligned} f(i \cdot 1) &= f(i) = -i \\ i f(1) &= i \cdot 1 = i \end{aligned}\right\} \text{different}$$