Lecture 6

Friday, January 17, 2020

We know that a linear map $f: \mathbb{R}^n \to \mathbb{R}^m$ can be represented by an $m \times n$ matrix with real coefficient. In particular, if f is represented by matrix A then

$$f(x) = \underbrace{Ax}_{\text{multiplication of matrix and column vector}}$$

We can ask if a linear map $f: V \to W$, where V and W are vector spaces over F, is associated with a matrix. If it is, how do we understand the product Ax (now that x is a vector of a general vector space V) ?

Recall that $\mathbb{R}^n$ has a natural basis called the standard basis

$$B_0 = \{e_1, e_2, \dots, e_n\}$$

If we choose a different basis $B_1 = \{v_1, v_2, \dots, v_n\}$ of $\mathbb{R}^n$, every vector in $\mathbb{R}^n$ has a coordinate vector with respect to B_1, denoted by $[v]_{B_1}$:

$$[v]_{B_1} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

means $v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$.

* <u>Definition</u>:

Let V be a vector space over F and $B = \{v_1, v_2, \dots, v_n\}$ be a basis of V. For each vector $v \in V$, we know that v can be written as a linear combination of $v_1, v_2, \dots, v_n$.

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

The vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is called the coordinate vector of v with respect to basis B.

This vector is denoted by $[v]_B$.

Ex: Vector space $V = \mathcal{P}_2(\mathbb{R})$ has basis $B_1 = \{x^2, x, 1\}$. The coordinate vector of $u(x) = 1 + 2x - x^2$ is

$$[u]_{B_1} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

If we pick a different basis, say $B_2 = \{1, 1+x, 1+x+x^2\}$ then what is the coordinate vector of f with respect to B_2?

We need to find $a, b, c \in \mathbb{R}$ such that

$$u = a.1 + b.(1+x) + c(1+x+x^2)$$

This equation is equivalent to

$$1 + 2x - x^2 = (a+b+c) + (b+c)x + cx^2$$

This is the equation of functions (a function equal to a function) Thus, the equality must be true for all $x \in \mathbb{R}$ We get

$$\begin{cases} 1 = a+b+c \\ 2 = b+c \\ -1 = c \end{cases}$$

which gives $a = -1$, $b = 3$, $c = -1$ Therefore,

$$[u]_{B_2} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

* Matrix representation of a linear map:

Let V and W be vector spaces over F and $f: V \to W$ be a linear map

Let $B_1 = \{v_1, v_2, \ldots, v_n\}$ be a basis of V and $B_2 = \{w_1, w_2, \ldots, w_m\}$ be a basis of W.

Then the matrix representation of f with respect to bases B_1 and B_2 is defined as

$$\begin{bmatrix} | & | & & | \\ [f(v_1)]_{B_2} & [f(v_2)]_{B_2} & \cdots & [f(v_n)]_{B_2} \\ | & | & & | \end{bmatrix}$$

and is denoted by $[f]_{B_2, B_1}$.

We get the following rule:

$$\underbrace{[f(v)]_{B_2}}_{m\times 1} = \underbrace{[f]_{B_2,B_1}}_{m\times n}\ \underbrace{[v]_{B_1}}_{n\times 1} \qquad \forall\, v \in V$$

$\underline{\underline{Ex}}$: let $f: \mathbb{C}^3 \longrightarrow M_{2\times 2}(\mathbb{C})$

$$f(a,b,c) = \begin{bmatrix} ai & b \\ (a+c)i & 0 \end{bmatrix}$$

Find a matrix representation of f.

First, we need to choose a basis for $\mathbb{C}^3$ and a basis for $M_{2\times 2}(\mathbb{C})$

Let's choose the standard bases.

$$B_1 = \{ \underbrace{(1,0,0)}_{e_1}, \underbrace{(0,1,0)}_{e_2}, \underbrace{(0,0,1)}_{e_3} \}$$

$$B_2 = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_4} \right\}$$

We have

$$f(e_1) = f(1,0,0) = \begin{bmatrix} i & 0 \\ i & 0 \end{bmatrix} = iE_1 + iE_3.$$

Thus,

$$[f(e_1)]_{B_2} = \begin{bmatrix} i \\ 0 \\ i \\ 0 \end{bmatrix}.$$

Similarly, we can find $[f(e_2)]_{B_2}$ and $[f(e_3)]_{B_2}$. We get

$$[f]_{B_2,B_1} = \begin{bmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ i & 0 & i \\ 0 & 0 & 0 \end{bmatrix}.$$

Ex: consider the derivative map (operator):

$$D: P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$$

given by $D(u) = u'$ Find a matrix representation of D.

First, we need to choose a basis for $P_3(\mathbb{R})$ and a basis for $P_2(\mathbb{R})$. Let's choose the standard bases:

$$B_1 = \{x^3, x^2, x, 1\},$$
$$B_2 = \{x^2, x, 1\}.$$

We have

$$D(x^3) = 3x^2.$$

Thus,

$$[D(x^3)]_{B_2} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

Similarly, $D(x^2) = 2x$, $D(x) = 1$, $D(1) = 0$. Thus,

$$[D(x^2)]_{B_2} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad [D(x)]_{B_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [D(1)]_{B_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We conclude that

$$[f]_{B_2, B_1} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

* Note: if $f: V \to W$ is linear and $\dim V = n$, $\dim W = m$ then $[f]_{B_2, B_1}$ is of size $m \times n$

Recall from Linear Algebra that composition of linear maps corresponds to multiplication of matrices. This principle is also true for linear maps on general vector spaces. Let's consider vector spaces U, V, W.

Suppose $f: U \to V$ and $g: V \to W$ are linear maps.

Let B_1 be a basis of U,
B_2 " " V,
B_3 " " W

Then we know that

$$[f(u)]_{B_2} = [f]_{B_2,B_1} [u]_{B_1}, \qquad \forall u \in U$$

$$[g(v)]_{B_3} = [g]_{B_3,B_2} [v]_{B_2} \qquad \forall v \in V$$

In the second equation, let us set $v = f(u)$. Then

$$[g(f(u))]_{B_3} = [g]_{B_3,B_2} [f(u)]_{B_2}$$
$$= [g]_{B_3,B_2} [f]_{B_2,B_1} [u]_{B_1}$$

This identity shows that

$$[g \circ f]_{B_3,B_1} = [g]_{B_3,B_2} [f]_{B_2,B_1}$$

In other words, the matrix representing a composite map is equal to the product of the matrices representing each linear map

* Null space of a linear map.

Let $f: V \to W$ be a linear map. One can define the null space (also called kernel) of f, like in Linear Algebra I, as follows.

$$\text{null}(f) = \{v \in V : f(v) = 0\}.$$

This is a subspace of V.

Ex:

$$f : M_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}^2$$

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b, c+d)$$

1) Find a matrix representation of f.

2) Find a basis and the dimension of null(f).

1) To find a matrix representation of f, we need to choose a basis for $V = M_{2\times 2}(\mathbb{R})$ and a basis for $W = \mathbb{R}^2$

Let us choose the standard bases for simplicity.

$$B_1 = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_4} \right\},$$

$$B_2 = \{ \underbrace{(1,0)}_{w_1}, \underbrace{(0,1)}_{w_2} \}$$

Recall that
$$[f]_{B_2,B_1} = \begin{bmatrix} | & | & | & | \\ [f(E_1)]_{B_2} & [f(E_2)]_{B_2} & [f(E_3)]_{B_2} & [f(E_4)]_{B_2} \\ | & | & | & | \end{bmatrix}$$

We have
$$f(E_1) = f\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = (1+0, 0+0) = (1,0) = w_1 = 1w_1 + 0w_2$$

Similarly,
$$f(E_2) = w_1 = 1w_1 + 0w_2$$
$$f(E_3) = w_2 = 0w_1 + 1w_2$$
$$f(E_4) = w_2 = 0w_1 + 1w_2$$

Thus,
$$[f]_{B_2,B_1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

2) Let us recall the definition of null space.

$$\text{null}(f) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : (a+b, c+d) = (0,0) \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : a+b = c+d = 0 \right\}$$

One can rewrite null(f) as

$$\text{null}(f) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ b = -a, \ d = -c \right\}$$

We will remove the constraints by substituting b by $-a$, and d by $-c$. After the substitution, there is no more constraint, i.e. a and c are independent variables.

$$\text{null}(f) = \left\{ \begin{bmatrix} a & -a \\ c & -c \end{bmatrix} . \ a, c \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} : \ a, c \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}}_{A_2} \right\}$$

We see that $\text{null}(f) = \text{span}\{A_1, A_2\}$. To say that $\{A_1, A_2\}$ is a basis of null(f), we need to show that A_1 and A_2 are linearly independent. Let us consider the equation

$$c_1 A_1 + c_2 A_2 = 0$$

with unknowns $c_1, c_2 \in \mathbb{R}$ This equation is equivalent to

$$c_1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which can be written as

$$\begin{bmatrix} c_1 & -c_1 \\ c_2 & -c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We get $c_1 = c_2 = 0$.

We conclude that $\{A_1, A_2\}$ is a basis of null(f) and

$$\dim \text{null}(f) = 2$$