Lecture 7

Wednesday, January 22, 2020

Recall that each linear map can be represented by a matrix.
Given a linear map $f: V \to W$, to represent f by a matrix, one has to choose a basis of V and a basis of W.

Suppose V has basis $B_1 = \{v_1, v_2, \dots, v_n\}$,
W has basis $B_2 = \{w_1, w_2, \dots, w_m\}$.

Then the matrix representing f is

$$[f]_{B_2,B_1} = \begin{bmatrix} | & | & & | \\ [f(v_1)]_{B_2} & [f(v_2)]_{B_2} & \cdots & [f(v_n)]_{B_2} \\ | & | & & | \end{bmatrix}$$

Matrix representation of a linear map provides a systematic way to do calculation on linear maps through matrices. In fact, Matlab is a software that regards every quantity as a matrix. (A number is considered as a 1×1 matrix.)

The addition and scaling of linear maps are compatible with the addition and scaling of matrices:

$$[f+g]_{B_2,B_1} = [f]_{B_2,B_1} + [g]_{B_2,B_1},$$
$$[cf]_{B_2,B_1} = c[f]_{B_2,B_1}.$$

The set of all linear maps from V to W is denoted by $\mathcal{L}(V,W)$ This is a subspace of W^V (the space of all functions from V to W)

The composition of linear maps is compatible with the multiplication of matrices: let $g: U \to V$ and $f: V \to W$ be linear maps. Let B_1, B_2, B_3 be bases of U, V, W respectively Then

$$\begin{array}{ccccc} U & \xrightarrow{g} & V & \xrightarrow{f} & W \\ B_1 & & B_2 & & B_3 \end{array}$$

Then the matrix that represents the composite map $f\circ g$ is

$$[f\circ g]_{B_3,B_1} = [f]_{B_3,B_2}\,[g]_{B_2,B_1}.$$

Given a linear map $f: V \to W$, the null space of f is defined as the set

$$\text{null}(f) = \{ v \in V : f(v) = 0 \}.$$

This is also a subspace of V. Why? Check carefully the following:

$$\begin{cases} \text{null}(f) \text{ is a subset of } V, \\ 0 \in \text{null}(f), \\ \text{null}(f) \text{ is closed under addition,} \\ \text{null}(f) \text{ is closed under scaling.} \end{cases}$$

The dimension of null(f) is called the nullity of f.

Note the similarity between the nullity of a linear map and the nullity of a matrix. If A is an $m \times n$ matrix then the null space of A is

$$\text{null}(A) = \{ v \in \mathbb{R}^n : Av = 0 \}$$

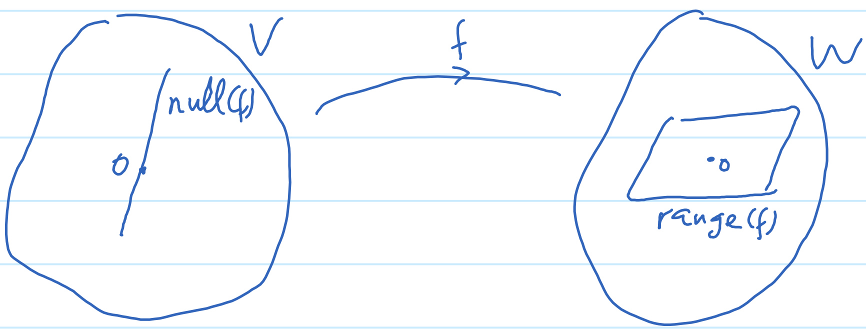
The range of f is defined as the set of all outputs of f:

$$\text{range}(f) = \{ f(v) : v \in V \}$$

This is a subspace of W. Why? Check carefully the following:

$$\begin{cases} \text{range}(f) \text{ is a subset of } W, \\ 0 \in \text{range}(f), \\ \text{range}(f) \text{ is closed under addition,} \\ \text{range}(f) \text{ is closed under scaling.} \end{cases}$$

The dimension of range(f) is called the rank of f.



Here is the connection between f and its matrix representation $A = [f]_{B_2, B_1}$:

- $\text{null}(f) = \{v \in V : f(v) = 0\}$ – set of all vectors that get mapped to 0.
- $\text{null}(A) = \{[v]_{B_1} \in \mathbb{R}^n : f(v) = 0\}$ – set of all <u>coordinates</u> of vectors that get mapped to 0.
- $\text{range}(f) = \{f(v) : v \in V\}$ – set of all images (outputs) of f.
- $\text{col}(A) = \{A[v]_{B_1} : v \in V\}$
  $= \{[f(v)]_{B_2} : v \in V\}$ – set of all <u>coordinates</u> of images of f.

- nullity of f = nullity of A,
- rank of f = rank of A

A linear map $f: V \to W$ is said to be a
- monomorphism if $\text{null}(f) = \{0\}$,
- epimorphism if $\text{range}(f) = W$,
- isomorphism if f is both monomorphic and epimorphic.

Another term for monomorphic is one-to-one or injective.
Another term for epimorphic is onto or surjective.
Another term for isomorphic is bijective.

<u>Ex</u>: Consider a linear map $G: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by

$$G(u) = xu' - u$$

(a) Find a basis of $\text{null}(G)$ what is the nullity of G? Is G a monomorphism?

(b) Find a basis of $\text{range}(G)$ what is the rank of G? Is G an epimorphism?

By the definition of null space,

$$\begin{aligned}\text{null}(G) &= \{u \in P_2 : G(u) = 0\}\\ &= \{u = ax^2+bx+c . \quad xu' - u = 0\}\\ &= \{u = ax^2+bx+c : \quad x(2ax+b) - (ax^2+bx+c) = 0\}\\ &= \{u = ax^2+bx+c : \quad ax^2 - c = 0\}\end{aligned}$$

Let us examine each element of null(G). For each $u \in$ null(G), we can write $u = ax^2+bx+c$ where $a,b,c \in \mathbb{R}$. The condition a,b,c have to satisfy is

$$ax^2 - c = 0 \quad \forall x \in \mathbb{R}.$$

By plugging $x=0$, we get $c=0$ Then plugging $x=1$ gives $a=0$. We see that there is no constraint on b. Thus,

$$\begin{aligned}\text{null}(G) &= \{u = bx \quad b \in \mathbb{R}\}\\ &= \text{span}\{x\}.\end{aligned}$$

Therefore, $\{x\}$ is a basis of null(G). The nullity of G is therefore 1 Because null$(G) \neq \{0\}$, G is not a monomorphism.

By the definition of the range space, we have

$$\begin{aligned}\text{range}(G) &= \{G(u) : u \in P_2\}\\ &= \{xu' - u . \quad u \in P_2\}\\ &= \{x(2ax+b) - (ax^2+bx+c) \quad : \quad u = ax^2+bx+c \in P_2\}\\ &= \{ax^2 - c \quad : \quad a,b,c \in \mathbb{R}\}\\ &= \text{span}\{x^2, 1\}.\end{aligned}$$

The set $\{x^2, 1\}$ is linearly independent because it is a subset of a linearly independent set $\{x^2, x, 1\}$ (the standard basis)

Therefore, $\{x^2, 1\}$ is a basis of range(G) The rank of G is therefore equal to 2. Since $P_2(\mathbb{R})$ is 3-dimensional, range(G) is not equal to $P_2(\mathbb{R})$. Thus, G is not an epimorphism