Lecture 8

Friday, January 24, 2020

Let us continue the example last time:

$$G : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$$
$$G(u) = (x+1)u'$$

Find a basis and the dimension of range(G)

By definition of the range space,

$$\begin{aligned}\text{range}(G) &= \{ G(u) : u \in P_2(\mathbb{R}) \} \\ &= \{ (x+1)u' : u = ax^2+bx+c,\ a,b,c \in \mathbb{R} \} \\ &= \{ (x+1)(2ax+b) : a,b,c \in \mathbb{R} \} \\ &= \{ 2a(x^2+x) + b(x+1) : a,b \in \mathbb{R} \} \\ &= \text{span}\{ \underbrace{x^2+x}_{u_1}, \underbrace{x+1}_{u_2} \}.\end{aligned}$$

To conclude that $\{u_1, u_2\}$ is a basis of range(G), we need to check if it is linearly independent. Consider the equation

$$c_1(x^2+x) + c_2(x+1) = 0$$

with unknowns $c_1, c_2 \in \mathbb{R}$. The equation has to be true for all $x \in \mathbb{R}$. One can see that there are infinitely many equations while only two unknowns. It would not be surprising that c_1 and c_2 have to be equal to zero. Pick $x = 1$, we get

$$2c_1 + 2c_2 = 0$$

or simply $c_1 + c_2 = 0$. Pick $x = 0$, we get $c_2 = 0$. Therefore, $c_1 = c_2 = 0$. We conclude that $\{u_1, u_2\}$ is indeed a basis of range(G).

$$\text{rank}(G) = 2$$

* Relation of null space and range space:

Let $f : V \longrightarrow W$ be a linear map. Let

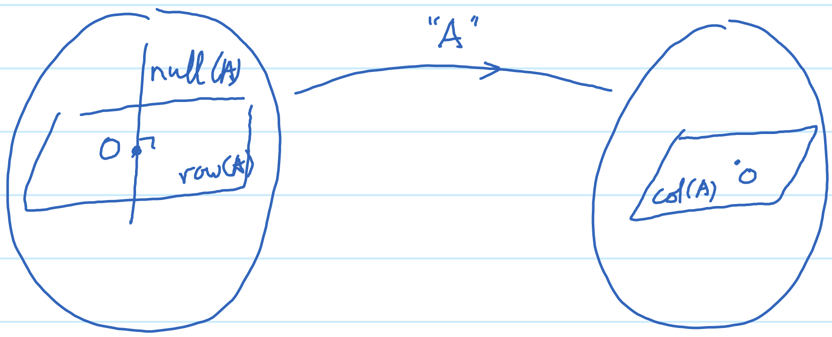
$$B_1 = \{v_1, v_2, \dots, v_n\},$$
$$B_2 = \{w_1, w_2, \dots, w_m\}$$

be bases of V and W respectively

The matrix representation of f is $A = [f]_{B_2, B_1}$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ [f(v_1)]_{B_2} & [f(v_2)]_{B_2} & \cdots & [f(v_n)]_{B_2} \\ | & | & & | \end{bmatrix}}_{m \times n}$$

Matrix A can be regarded as a linear map "A" from $\mathbb{R}^n$ to $\mathbb{R}^m$ which does the following: it maps each $x \in \mathbb{R}^n$ to vector $Ax \in \mathbb{R}^m$.



The column space of matrix A is defined as the space spanned by the columns of A. If we write

$$A = \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}$$

Then $col(A) = span\{c_1, c_2, \dots, c_n\}$

$$= \{a_1 c_1 + a_2 c_2 + \dots + a_n c_n : a_1, a_2, \dots, a_n \in \mathbb{R}\}$$

Note that one can write

$$a_1 c_1 + a_2 c_2 + \dots + a_n c_n = \underbrace{\begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{x} = Ax$$

Thus, $\quad \mathrm{col}(A) = \{Ax : x \in \mathbb{R}^n\}$.

This is exactly the range of the linear map "A". The column space of A, understood as a matrix, is equal to the range of A, understood as a linear map.

How about the row space of A? The row space is contained in the domain ($\mathbb{R}^n$), while the column space is contained in the target set ($\mathbb{R}^m$).

$$A = \begin{bmatrix} \text{——} & R_1 & \text{——} \\ \text{——} & R_2 & \text{——} \\ & \vdots & \\ \text{——} & R_m & \text{——} \end{bmatrix}$$

By definition of row space,

$$\mathrm{row}(A) = \mathrm{span}\{R_1, R_2, \ldots, R_m\}$$

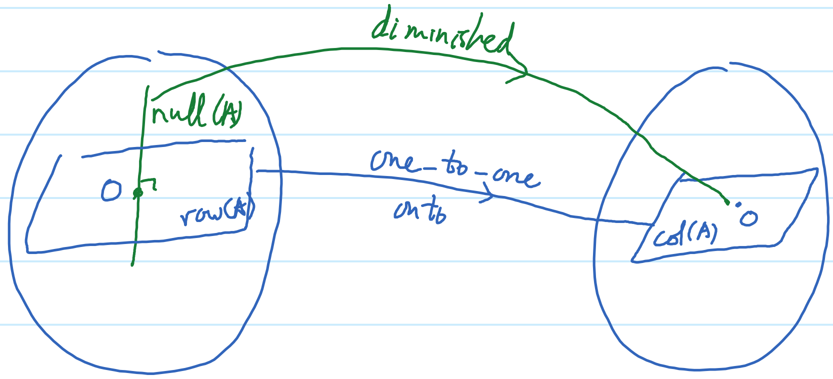
We will see that the row space is perpendicular to the null space of A. To see this, we take an $x \in \mathrm{null}\, A$. We know that $Ax = 0$. In other words,

$$\underbrace{\begin{bmatrix} \text{——} & R_1 & \text{——} \\ \text{——} & R_2 & \text{——} \\ & \vdots & \\ \text{——} & R_m & \text{——} \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} R_1 \cdot x \\ R_2 \cdot x \\ \vdots \\ R_m \cdot x \end{bmatrix}$$

Hence, $R_1 \cdot x = R_2 \cdot x = \ldots = R_m \cdot x = 0$. In other words, x is pendicular to vectors $R_1, R_2, \ldots, R_m$. Thus, x is perpendicular to $\mathrm{row}(A)$. Because x was taken arbitrarily in $\mathrm{null}(A)$, we conclude that

$$\mathrm{null}\, A \perp \mathrm{row}\, A.$$



Every vector on null A will be mapped to 0. (i.e null A is collapsed by A). If the linear map A is restricted on row A then the map will be one-to-one and onto. The row space and the column space are of the same "size". To be more specific,

$$\dim \text{row}(A) = \dim \text{col}(A).$$

This number is called the rank of A.

There is an important result in Linear Algebra called rank-nullity theorem. Recall the rank-nullity theorem for matrices states that:

Let A be a matrix (not necessarily square). Then

$$\text{rank}(A) + \text{nullity}(A) = \#\text{ columns of } A.$$

One can translate this theorem in terms of linear maps as follows.

Let $f: V \to W$ be a linear map. Then

$$\text{rank}(f) + \text{nullity}(f) = \dim V.$$

This theorem gives a way to compute rank through nullity and vice versa. If the rank if large (f is "rich" in values) then the nullity is small (f diminishes less).

Recall that the rank of a matrix is the number of nonzero rows (or equivalently, the number of pivot columns) in its RREF.

One can compute the rank and nullity through RREF. For example,

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{bmatrix}$$

On Matlab:

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>> A = [1 2 3 0; 4 5 6 0; 7 8 9 0]
>> rref(A)
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The output is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that $\text{rank}(A) = 2$. By rank-nullity theorem,

$$\text{nullity}(A) = 4 - \text{rank}(A) = 2.$$