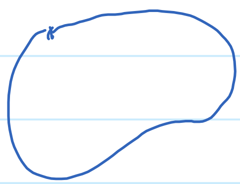
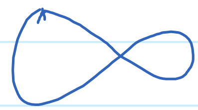
Lecture 9

Monday, January 27, 2020

A curve on the 2-dimensional plane can be regarded as a continuous map $f: \mathbb{R} \to \mathbb{R}^2$. There is a curve that fills the entire plane, known as a space-filling curve. This is an interesting object in a field of topology. A space-filling curve is an onto continuous map from $\mathbb{R}$ to $\mathbb{R}^2$. It is hard to visualize such a curve because it is pathological in many ways. For example, the curve is not a simple curve. In fact, it intersects itself infinitely many times.



simple curve



not a simple curve

However, one cannot find any map $f: \mathbb{R} \to \mathbb{R}^2$ that is both linear and onto (epimorphic). This will be explained by the rank-nullity theorem.

Let $f: V \to W$ be a linear map. The rank-nullity theorem says that

$$\operatorname{rank}(f) + \operatorname{nullity}(f) = \dim V.$$

Consequently, the following statements are equivalent:

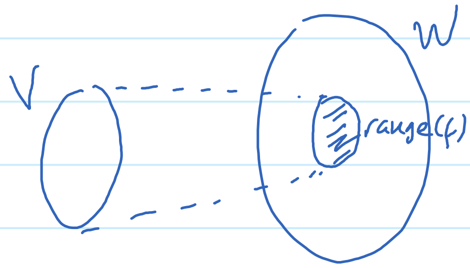
- f is monomorphic
- $\operatorname{nullity}(f) = 0$.
- $\operatorname{rank}(f) = \dim V$.

The following statements are also equivalent:

- f is epimorphic.
- $\operatorname{rank}(f) = \dim W$.
- $\operatorname{nullity}(f) = \dim V - \dim W$

Let us consider 3 situations:

- $\dim V < \dim W$:

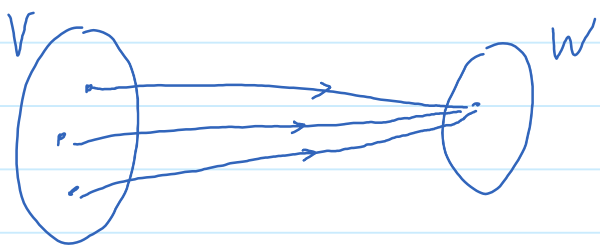


In this case, f maps a small vector space into a large vector space. Intuitively, one can expect that f is not epimorphic. This can be proved as follows:

$$\begin{aligned} \dim \operatorname{range}(f) = \operatorname{rank}(f) &= \dim V - \operatorname{nullity}(f) \\ &\le \dim V \\ &< \dim W. \end{aligned}$$

Thus, range(f) is strictly smaller than W. Therefore, f is not epimorphic.
In particular, there is no linear map $f: \mathbb{R} \to \mathbb{R}^2$ that is onto.
"One cannot cover a big bed with a small blanket"

- $\dim V > \dim W$.



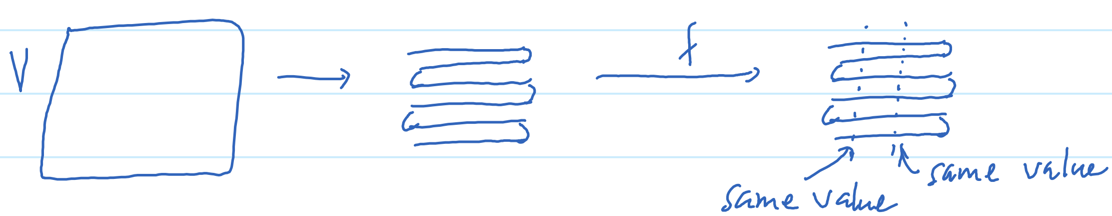
In this case, f maps a big space into a smaller vector space. One can expect that f is not one-to-one. This observation can be proved as follows:

$$\operatorname{rank}(f) \le \dim W < \dim V.$$

Thus,

$$\operatorname{nullity}(f) = \dim V - \operatorname{rank}(f) \ge 1.$$

"One cannot fit a big blanket into a suitcase without folding it first."



■ $\dim V = \dim W$:

Put $n = \dim V = \dim W$ By rank-nullity theorem, we observe that

$$\underbrace{\text{nullity}(f) = 0}_{f \text{ is monomorphic}} \iff \underbrace{\text{rank}(f) = n}_{f \text{ is epimorphic}} \iff f \text{ is isomorphic.}$$

Thus, in this case, "monomorphic", "epimorphic", and "isomorphic" are the same.

We summarize our above observations as follows:

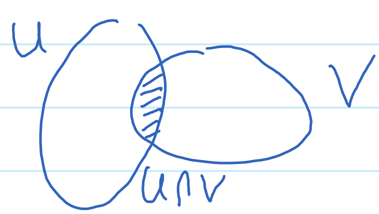
* Theorem:

Let $f: V \to W$ be a linear map.

- If $\dim V < \dim W$ then f is not monomorphic
- If $\dim V > \dim W$ then f is not epimorphic.
- If $\dim V = \dim W$ then f is monomorphic if and only if it is epimorphic.

<u>Sum of two vector spaces</u>

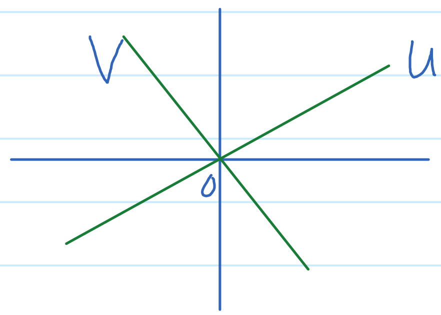
Consider two vector spaces U and V (over the same field F). It is easy to check that $\underbrace{U \cap V}_{\text{intersection of } U \text{ and } V}$ is also a vector space.



How to check? Observe that $U \cap V$ is a subset of U, which is a vector space One only needs to check 3 properties:

exercise
(1) $0 \in U \cap V$,
(2) $U \cap V$ is closed under addition,
(3) $U \cap V$ is closed under scaling.

Ex: In $\mathbb{R}^2$, consider two lines passing through the origin.

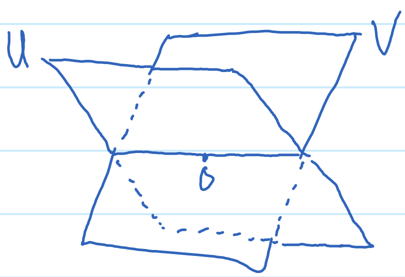


Each line can be viewed as a 1-dimensional vector space. We see from the picture that

$$U \cap V = \{0\}$$

which is a vector space

Ex: In $\mathbb{R}^3$, consider two planes that pass through the origin.

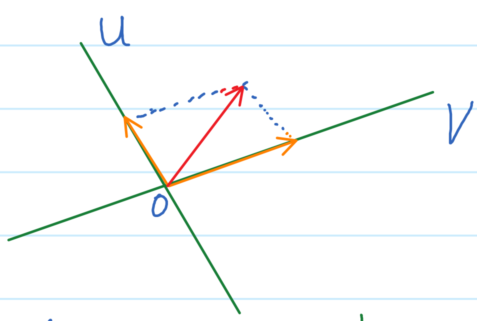


The intersection of these plane is a line passing through the orgin. Thus, $U \cap V$ is also a vector space.

While the intersection $U \cap V$ is always a vector space, the union $U \cup V$ is generally not a vector space. A simple example is that:

U and V are two lines on the plane.

The union of two lines is not a vector space because it is not closed under addition.

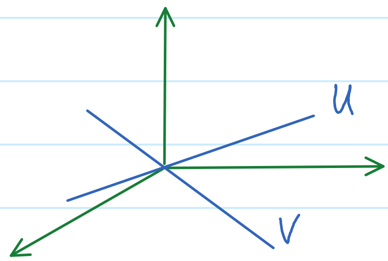


The red vector is outside of the union.

A natural question is: what is a vector space that contains both U and V?

The union $U \cup V$ is not an answer because it is not a vector space.

There are in fact infinitely many vector spaces that contain both U and V. For example, when U and V are lines, the plane that contains both lines is such a vector space. The 3-dimensional space as in the picture is also a vector space that contains both U and V.



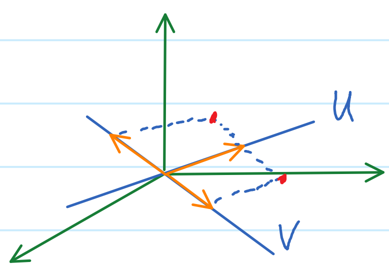
We therefore adjust the question to make it more meaningful:

What is the smallest vector space that contains both U and V?

This vector space will be denoted as $U+V$ (the sum of vector space U and vector space V). It is defined as

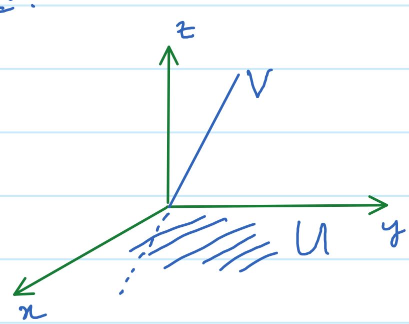
$$U+V=\{u+v : u\in U,\ v\in V\}.$$

Ex:



$U+V$ = plane that contains both lines U and V.

Ex:



U = xy-plane

V = line passing through the origin

$U+V=\mathbb{R}^3$.

How to find a basis of $U+V$?

Let B_1 be a basis of U, and B_2 be a basis of V. We know that

$$U = \operatorname{span} B_1 \quad \text{and} \quad V = \operatorname{span} B_2$$

Write

$$B_1 = \{v_1, v_2, \ldots, v_n\},$$

$$B_2 = \{w_1, w_2, \ldots, w_m\}.$$

The concatenation of B_1 and B_2 is defined as

$$B_1 \sqcup B_2 = \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m\}$$

There is a slight difference between concatenation and union. If there are common vectors between B_1 and B_2 then the union $B_1 \cup B_2$ includes each of these vectors only once, while the concatenation includes these vectors twice For example,

$$B_1 = \{1, 2, 3\}$$

$$B_2 = \{3, 4, 5\}$$

$$B_1 \cup B_2 = \{1, 2, 3, 4, 5\}$$

$$B_1 \sqcup B_2 = \{1, 2, 3, 3, 4, 5\}.$$

We see that the span of $B_1 \sqcup B_2$ includes both U and V. Moreover, any vector space that contains both U and V must also contain $B_1 \sqcup B_2$. Thus, $\operatorname{span}(B_1 \sqcup B_2)$ is the smallest vector space that contains both U and V. We get

$$U + V = \operatorname{span}(B_1 \sqcup B_2)$$

To find a basis of $U+V$, we only need to extract linearly independent vectors from $B_1 \sqcup B_2$

We do so by arranging vectors $v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m$ as columns of a matrix.

$$\begin{bmatrix} | & | & & | & | & | & & | \\ v_1 & v_2 & \cdots & v_n & w_1 & w_2 & \cdots & w_m \\ | & | & & | & | & | & & | \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

pivot columns tell us which vectors among $v_1, v_2, \ldots, w_m$ to keep.

We will consider some examples next time.