MATH 342, MIDTERM EXAM, WINTER 2020

Name	Student ID

- Answer to each problem must be written coherently in full sentences. Answers not supported by valid arguments will not receive full credit. Start a sentence with words rather than a formula. Use words to transition your ideas, for example "This leads to", "Therefore", "We want to show", etc.

- Read carefully the description of each problem. Make sure that you do all parts of the problem. ~~You have 2 pages to work on Problem 2, Part (b)~~. You can use the blank page on the back if need more space

- Doing correctly Problems 2, 3, 4, 5 will grant you 100% credit of the exam. Each correct answer in Problem 1 will give you one bonus point.

Problem	Possible points	Earned points
1	7	
2a	10	
2b	10	
3	10	
4	10	
5	10	
Total	57	

Problem 1. (7 points) To each of the following statements, circle T if the statement is true. Circle F if it is false.

1. (T / F) The set $\{u \in P_2(\mathbb{R}) : \ u(0) = 1\}$ is a vector space over $\mathbb{R}$.

2. (T / F) There is an onto linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$.

3. (T / F) There is an onto linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$.

4. (T / F) If $\dim U = 3$ and $\dim V = 2$ then $\dim(U + V)$ can never exceed 5.

5. (T / F) The dimension of the complex vector space of complex polynomials of degree at most n is equal to n.

6. (T / F) If $B = \{v_1, v_2, \ldots, v_n\}$ is linearly independent then any nonempty subset of B is also linearly independent.

7. (T / F) If S is a finite set of vectors that span a vector space V then there exists a subset of S that is a basis of V.

Problem 2. Let $V = \{A \in M_{2\times 2}(\mathbb{R}) : A = -A^T\}$. Here A^T denotes the transpose of matrix A.
(a) (10 points) Show that V is a vector space over $\mathbb{R}$.

By the definition of V, V is a subset of $M_{2\times2}(\mathbb{R})$, which is known to be a vector space over $\mathbb{R}$. Thus, we only need to show 3 things:

1) $0 \in V$
2) V is closed under addition.
3) V is closed under scaling.

Show 1): The zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ belongs to V because $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T$.

Show 2): Let $A, B \in V$. We show that $A+B \in V$.

For that, we need to show

$$A+B = -(A+B)^T. \qquad (*)$$

We know that

$$RHS(*) = -(A^T+B^T) = -A^T - B^T$$

Because $A, B \in V$, we have $A^T = -A$ and $B^T = -B$. Thus,

$$RHS(*) = A+B = LHS(*).$$

Therefore, V is closed under addition.

Show 3 Let $c \in \mathbb{R}$ and $A \in V$. We show that $cA \in V$.

That is to show $cA = -(cA)^T$. We have

$$(cA)^T = cA^T \underset{\uparrow}{=} -cA$$

because $A \in V$

Thus, V is closed under scaling

(b) (10 points) Find a basis of V. What is the dimension of V?

We can write V as

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\begin{bmatrix} a & c \\ b & d \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : a=-a,\ b=-c,\ c=-b,\ d=-d \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : a=d=0,\ b=-c \right\}$$

$$= \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \in \mathbb{R} \right\}$$

$$= \left\{ b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} : b \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

Thus, $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ is a basis of V.

$\dim V = 1$.

(Extra space for Problem 2b.)

Problem 3. (10 points) Consider a linear map $G : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $G(u) = u + u'$. Find a matrix representation of G.

We choose a basis on $P_2(\mathbb{R})$: $B = \{\underbrace{x^2}_{v_1}, \underbrace{x}_{v_2}, \underbrace{1}_{v_3}\}$ (the standard basis)

By the definition of $[G]_{B,B}$, we have

$$[G]_{B,B} = \begin{bmatrix} | & | & | \\ [G(v_1)]_B & [G(v_2)]_B & [G(v_3)]_B \\ | & | & | \end{bmatrix}.$$

We have

$$G(v_1) = v_1' + v_1 = 2x + x^2 = 2v_2 + v_1$$

Then $$[G(v_1)]_B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

Similarly, $$G(v_2) = v_2' + v_2 = 1 + x = v_3 + v_2.$$

Then $$[G(v_2)]_B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Also, $$G(v_3) = v_3' + v_3 = 0 + 1 = 1 = v_3.$$

Then $$[G(v_3)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In conclusion, the matrix representing G in basis B is

$$[G]_{B,B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Problem 4. (10 points) Show that the map G given in Problem 3 is a monomorphism.

To show that G is a monomorphism, we show that $\mathrm{null}(G) = \{0\}$.

Let $u \in \mathrm{null}(G)$. We can write $u = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$.

Because $u \in \mathrm{null}(G)$, $G(u) = 0$. In other words, $u' + u = 0$.

Thus, $(2ax + b) + (ax^2 + bx + c) = 0$.

Equivalently, $ax^2 + (2a+b)x + b + c = 0$

This is true for all $x \in \mathbb{R}$ only if $a = 2a+b = b+c = 0$.

This gives $a = b = c = 0$. Thus, $u = 0$. We have showed that $\mathrm{null}(G) = \{0\}$

Therefore, G is a monomorphism.

Problem 5. (10 points) Consider two vector spaces

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ x_1 = x_2 = x_3\},$$
$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ x_1 = -x_3, \ 2x_2 = x_4\}.$$

~~Find a basis of $U + V$. Is $U + V$ a direct sum?~~ Show that $U+V$ is a direct sum.

We will show that $U \cap V = \{0\}$.

Each $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ satisfies the system

$$\begin{cases} x_1 = x_2 = x_3, \\ x_1 = -x_3, \ 2x_2 = x_4. \end{cases}$$

From here we get $x_1 = x_3$, $x_1 = -x_3$ This implies $x_1 = 0$. Thus, $x_2 = x_3 = 0$.

Then $x_4 = 0$ We conclude that $U \cap V = \{0\}$.