Worksheet

1/27/2020

Name: ________________________________

Consider two vector spaces

$$\begin{aligned} U &= \{f \in P_3(\mathbb{C}) : \ \ f(0) = f(1) = 0\}, \\ V &= \{f \in P_3(\mathbb{C}) : \ \ f(1) = f(i) = 0\}. \end{aligned}$$

1. Find a basis of U. What is the dimension of U?
2. Find a basis of V. What is the dimension of V?
3. Find a basis of $U + V$. What is the dimension of $U + V$?

1) Find a basis of U:

$$\begin{aligned} U &= \{ f(z) = az^3 + bz^2 + cz + d : \ f(0) = f(1) = 0 \} \\ &= \{ az^3 + bz^2 + cz + d : \ d = 0, \ a+b+c+d = 0 \} \\ &= \{ az^3 + bz^2 + cz : \ c = -a-b \} \\ &= \{ az^3 + bz^2 + (-a-b)z : \ a, b \in \mathbb{C} \} \\ &= \{ a(z^3 - z) + b(z^2 - z) . \ a, b \in \mathbb{C} \} \\ &= \operatorname{span}\{ \underbrace{z^3 - z}_{u_1}, \underbrace{z^2 - z}_{u_2} \} \end{aligned}$$

eliminate constraints

now a and b are free

We check that u_1 and u_2 are linearly independent consider the equation $c_1 u_1 + c_2 u_2 = 0$ with unknowns $c_1, c_2 \in \mathbb{C}$. We have

$$c_1(z^3 - z) + c_2(z^2 - z) = 0 \quad \forall z \in \mathbb{C}.$$

Pick $z = -1$ and $z = 2$, we get $c_1 = c_2 = 0$.

Therefore, $\{u_1, u_2\}$ is a basis of U

2) Find a basis of V:

$$V = \{ f(z) = az^3 + bz^2 + cz + d \quad f(1) = f(i) = 0$$

$$= \{ az^3 + bz^2 + cz + d \;.\; a + b + c + d = 0,\; ai^3 + bi^2 + ci + d = 0 \}$$

$$= \{ az^3 + bz^2 + cz + d \;:\; a + b + c + d = 0,\; -ai - b + ci + d = 0 \}$$

We solve a system of two equations for a, b, c, d. The matrix form is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -i & -1 & i & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 + iR_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1+i & 2i & 1+i \end{bmatrix}$$

$$\xrightarrow{R_2 = R_2/(-1+i)} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -i+1 & -i \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & i & 1+i \\ 0 & 1 & -i+1 & -i \end{bmatrix}$$

$$\begin{array}{cccc} & & \uparrow & \uparrow \\ & & \multicolumn{2}{c}{\text{free variables}} \end{array}$$

We get $\quad c = c, \quad d = d, \quad b = (i-1)c + id, \quad a = -ic - (1+i)d.$

Thus,

$$V = \{ (-ic - (1+i)d) z^3 + ((i-1)c + id) z^2 + cz + d \;:\; c, d \in \mathbb{C} \}$$

$$= \{ c(-iz^3 + (i-1)z^2 + z) + d(-(1+i)z^3 + iz^2 + 1) \;:\; c, d \in \mathbb{C} \}$$

$$= \operatorname{span} \{ \underbrace{-iz^3 + (i-1)z^2 + z}_{v_1},\; \underbrace{-(1+i)z^3 + iz^2 + 1}_{v_2} \}$$

One can check that v_1 and v_2 are linearly independent

3) Find a basis of $U+V$.

We notice that U and V are abstract vector spaces. For this reason, we will work on the coordinate vectors. U and V are subspaces of $P_3(\mathbb{C})$.

Let us choose the standard basis of $P_3(\mathbb{C})$, namely

$$B = \{z^3, z^2, z, 1\}$$

The coordinate vectors of u_1, u_2, v_1, v_2 in this basis are

$$u_1' = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad u_2' = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_1' = \begin{bmatrix} -i \\ i-1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2' = \begin{bmatrix} -1-i \\ i \\ 0 \\ 1 \end{bmatrix}$$

U corresponds to $U' = \operatorname{span}\{u_1', u_2'\}$

V corresponds to $V' = \operatorname{span}\{v_1', v_2'\}$

$$\begin{bmatrix} | & | & | & | \\ u_1' & u_2' & v_1' & v_2' \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & -i & -1-i \\ 0 & 1 & i-1 & i \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & -1-i & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

Thus, $\{u_1', u_2', v_2'\}$ is a basis of $U'+V'$. Therefore,

$\{u_1, u_2, v_2\}$ is a basis of $U+V$.

About dimension, $\dim(U+V) = 3$, $\dim U = 2$, $\dim V = 2$.