Worksheet

1/31/2020

Name: ____________________________________

1. Let $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_2 = x_1 + x_3,\ x_3 = 2x_1 - x_2 + 5x_4\}$. Find a subspace W of $\mathbb{R}^4$ such that $V \oplus W = \mathbb{R}^4$.

See Lecture 11

2. Consider two vector spaces

$$\begin{aligned} V_1 &= \{A \in M_{2\times 2}(\mathbb{R}): \quad A = A^T\}, \\ V_2 &= \{A \in M_{2\times 2}(\mathbb{R}): \quad A = -A^T\}. \end{aligned}$$

Show that $V_1 \oplus V_2 = M_{2\times 2}$.

First, we find a basis of V_1:

$$V_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \cdot \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \quad b = c \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} : \quad a, b, d \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : a, b, d \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{v_3} \right\}$$

To check if $\{v_1, v_2, v_3\}$ is a basis of V_1, we need to check if it is linearly independent. Let us consider the equation $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ with unknowns $c_1, c_2, c_3 \in \mathbb{R}$ This equation can be written as

$$\begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $c_1 = c_2 = c_3 = 0$. We conclude that $B_1 = \{v_1, v_2, v_3\}$ is a basis of V_1.

Similarly, one can show that

$$B_2 = \left\{ \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{v_4} \right\}$$

is a basis of V_2.

To show that $V_1 \oplus V_2 = M_{2\times 2}(\mathbb{R})$, we need to show 2 things:

- The sum $V_1 + V_2$ is a direct sum
- The sum $V_1 + V_2$ is 4-dimensional (and thus it must be equal to $M_{2\times2}(\mathbb{R})$).

Both statements can be showed at once by showing that $B_1 \sqcup B_2$ is linearly independent. We have

$$B_1 \sqcup B_2 = \{v_1, v_2, v_3, v_4\}.$$

One can show that these vectors are linearly independent directly or through coordinates. Let's use the second method. Consider the standard basis of $M_{2\times2}(\mathbb{R})$, namely

$$B = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_4} \right\}.$$

Then the coordinates of v_1, v_2, v_3, v_4 are

$$v_1' = [v_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad v_2' = [v_2]_B = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

$$v_3' = [v_3]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad v_4' = [v_4]_B = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

To check if v_1', v_2', v_3', v_4' are linearly independent, we arrange them into a matrix

$$\begin{bmatrix} | & | & | & | \\ v_1' & v_2' & v_3' & v_4' \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, $\{v_1', v_2', v_3', v_4'\}$ are linearly independent. Consequently, $\{v_1, v_2, v_3, v_4\}$ are also linearly independent