MTH 342 Worksheet 1

Name: **Answer Key** **Recitation time:** ____________

Show your work for each problem.

The answers given in this key are not written in full detail. These answers would NOT be sufficient for similar problems on your homework.

1. Determine which of the following sets of vectors are linearly dependent. If they are dependent, find a non-trivial linear combination of the vectors that is equal to the zero vector.

(a) $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}$

solution: Dependent:

$\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix} - \begin{bmatrix}0\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

(b) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}$

solution: Independent: one is not a scalar multiple of the other.

(c) $\begin{bmatrix}0\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$

solution: Independent:

$$\det\left(\begin{bmatrix}0 & 3 & 1\\2 & 1 & 0\\1 & 0 & 1\end{bmatrix}\right) = -7 \neq 0$$

2. Consider the basis

$$\mathcal{B} = \left\{\begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}-1\\3\end{bmatrix}\right\}$$

for $\mathbb{R}^2$. Find the coordinates of the vector $v = \begin{bmatrix}1\\3\end{bmatrix}$ with respect to the basis $\mathcal{B}$.

solution: Solve the system of equations:

$$c_1\begin{bmatrix}2\\4\end{bmatrix} + c_2\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix} \quad\to\quad \left[\begin{array}{cc|c}2 & -1 & 1\\4 & 3 & 3\end{array}\right] \quad\to\quad \left[\begin{array}{cc|c}1 & 0 & \frac{3}{5}\\0 & 1 & \frac{1}{5}\end{array}\right] \quad\to\quad [v]_{\mathcal{B}} = \begin{bmatrix}\frac{3}{5}\\\frac{1}{5}\end{bmatrix}$$

3. Consider the basis

$$\mathcal{B} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$$

for $\mathbb{R}^3$. Find the coordinates of the vector $v = \begin{bmatrix}-5\\2\\3\end{bmatrix}$ with respect to the basis $\mathcal{B}$.

solution:

$$[v]_{\mathcal{B}} = \begin{bmatrix}-7\\-1\\3\end{bmatrix}$$

Recall that a vector space is a set V and a field of scalars F with addition and scalar multiplication satisfying the following axioms:

(A0) Closed under addition: If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.

(A1) Commutativity of addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$.

(A2) Associativity of addition: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

(A3) Zero vector: There exists a vector $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.

(A4) Additive inverse: For each vector $\mathbf{v} \in V$ there exists a vector $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.

(S0) Closed under scalar multiplication: If $\mathbf{v} \in V$ and $\alpha \in F$, then $\alpha\mathbf{v} \in V$.

(S1) Multiplicative identity: $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.

(S2) Associativity of scalar multiplication: $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$ for all $\mathbf{v} \in V$ and all $\alpha, \beta \in F$.

(D1) Distribution over vector addition: $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$ and all $\alpha \in F$.

(D2) Distribution over scalar addition: $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$ for all $\mathbf{v} \in V$ and all $\alpha, \beta \in F$.

5. The following sets are not vector spaces over $\mathbb{R}$. Show that they violate at least one of the vector space axioms.

(a) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 = 1\}$
solution: Violates (A0), (A3), and (S0).

(b) The set $\mathbb{Z}$ of integers
solution: Violates (S0). Consider $\alpha = \frac{1}{2}$ and $v = 1$.

(c) The set $\mathbb{R}^2$ with scalar multiplication defined by $\lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ 0 \end{bmatrix}$ for all $\lambda \in \mathbb{R}$, $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

solution: Violates (S1). Consider $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) The set $\mathbb{R}^2$ with scalar multiplication defined by $\lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ y \end{bmatrix}$ for all $\lambda \in \mathbb{R}$, $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

solution: Violates (D2). Consider $\alpha = \beta = 1$ and $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

6. Find the eigenvalues and associated eigenvectors of the following matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

solution: The eigenvalues are the solutions to the equation $\det(A - \lambda I) = 0$. In this case the eigenvalues are $\lambda = 5$ and $\lambda = -1$.

Eigenvectors:

$$\lambda = 5: \qquad t\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}, \ t \neq 0$$

$$\lambda = -1: \qquad t\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}, \ t \neq 0$$