MTH 342 Worksheet 10

Name: Answer Key **Recitation time:** ________

Show your work for each problem.

1. Let $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$. Find the singular value decomposition of A.

Solution: We want to find unitary matrices P, Q and a diagonal matrix D such that

$$A = PDQ^*.$$

We start by finding the eigenvalues of A^*A. Notice that

$$A^*A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix},$$

The eigenvalues are $\lambda_1 = 8$ and $\lambda_2 = 2$. The Singular values are then

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{8} \qquad \text{and} \qquad \sigma_2 = \sqrt{\lambda_2} = \sqrt{2},$$

so the matrix D is given by

$$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix}.$$

Next we want to find the unitary matrix P. The columns of P are eigenvectors of

$$AA^* = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}.$$

Calculate the eigenvectors as you normally would. For $\lambda_1 = 8$ we get the eigenvector

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For $\lambda_2 = 2$ we get the eigenvector

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Since these are already unit vectors, they will be the columns of our unitary matrix P:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that if these had not been unit vectors, we would have divided each vector by its magnitude to get the columns of P.

We can find the columns of Q by using the formula

$$q_k = \frac{1}{\sigma_k} A^* p_k.$$

In this case

$$\begin{aligned} q_1 &= \frac{1}{\sqrt{8}} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{8}} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} q_2 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}. \end{aligned}$$

The matrix Q is then

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Note that we could have alternatively found Q first. In that case, Q would have columns that are an orthonormal set of eigenvectors for A^*A. The columns of P would be given by the formula

$$p_k = \frac{1}{\sigma_k} A q_k.$$

I choose to find P first simply because the calculations were a bit easier.

2. Find the plane of best fit $z = ax + by + c$ for the points $(1, 1, -1)$, $(2, 0, 1)$, $(0, 0, 5)$, $(0, 1, 1)$.

Solution: We want to minimize the quantity

$$\sum_{k=1}^{n} |ax_k + by_k + c - z_k|^2.$$

This is equivalent to finding the least-squares solution to the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 5 \\ 1 \end{bmatrix}.$$

To find the least-squares solution, we need to solve the equation $A^*A\vec{x} = A^*\vec{b}$. We calculate

$$A^*A = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}, \qquad A^*\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}.$$

We can now solve for $\vec{x}$ by reducing the augmented system:

$$\left[\begin{array}{ccc|c} 5 & 1 & 3 & -1 \\ 1 & 2 & 2 & 0 \\ 3 & 2 & 4 & 6 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Therefore

$$z = -2x - 4y + 5$$

is the plane of best fit for the given points.

3. Let V be a finite-dimensional inner product space and let $P : V \to V$ such that $P^2 = P$.

(a) Prove that if P is the orthogonal projection onto some subspace U of V, then P is self-adjoint.
(Hint: use the fact that $V = U \oplus U^\perp$)
Solution: To show P is self-adjoint, we want to show that $(Px, y) = (x, Py)$ for all $x, y \in V$. Since $V = U \oplus U^\perp$ we can write

$$x = u_1 + w_1, \qquad y = u_2 + w_2, \qquad u_1, u_2 \in U \quad w_1, w_2 \in U^\perp.$$

Note that

$$(u_1, w_2) = (w_1, u_2) = 0.$$

Then

$$\begin{aligned}
(Px, y) &= (u_1, y) \\
&= (u_1, u_2 + w_2) \\
&= (u_1, u_2) + (u_1, w_2) \\
&= (u_1, u_2) \\
&= (u_1, u_2) + (w_1, u_2) \\
&= (u_1 + w_1, u_2) \\
&= (x, u_2) \\
&= (z, Py).
\end{aligned}$$

(b) Prove that if P is self-adjoint, then P is the orthogonal projection onto $U = \text{range}(P)$.
(Hint: Prove $v - PV \in \ker(P)$. Consider $v = Pv + (v - PV)$ and use the fact that $\ker(P) = \text{range}(P^*)^\perp$)
Solution: We want to show that for any $v \in V$,

$$Pv = \text{proj}_U(v).$$

First, recall that $V = U \oplus U^\perp$, and if we write

$$v = u + w, \quad u \in U, \; w \in U^\perp,$$

then $\text{proj}_U(v) = u$,
Suppose P is self-adjoint, so $P = P^*$. Recall that $P^2 = P$, so

$$P(v - PV) = Pv - P^2 v = 0$$

and thus

$$v - Pv \in \ker(P) = \text{range}(P^*)^\perp = \text{range}(P)^\perp.$$

Naturally we can write v as

$$v = Pv + (v - Pv) = u + w$$

where $u \in U = \text{range}(P)$ and $w \in U^\perp = \ker(P)$. Therefore

$$\text{proj}_U(v) = u = Pv$$

as desired. Since v was arbitrary, this means that $P = \text{proj}_U$.