Worksheet

2/7/2020

Name: ____________________________________

1. Consider a linear map $f : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by $f(A) = A^T + A$. Find a matrix representation of f.

Choose the standard basis of $M_{2\times2}(\mathbb{R})$.

$$\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_4} \right\}.$$

We have

$$f(E_1) = f\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = 2E_1.$$

$$f(E_2) = f\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_2 + E_3.$$

$$f(E_3) = \cdots = E_2 + E_3$$

$$f(E_4) = \cdots = 2E_4.$$

Then

$$[f]_{\mathcal{B}} = \begin{bmatrix} | & | & | & | \\ [f(E_1)]_{\mathcal{B}} & [f(E_2)]_{\mathcal{B}} & [f(E_3)]_{\mathcal{B}} & [f(E_4)]_{\mathcal{B}} \\ | & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

2. Consider the linear map $G : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $G(u) = xu' - u$. Is G a monomorphism, epimorphism, isomorphism or none of them? Explain your answer.

Because the dimension of the domain of G is equal to the dimension of the target set of G (equal to 3), the properties of being monomorphic, epimorphic, isomorphic are equivalent In other words, if G is monomorphic then it is also epimorphic and isomorphic. If G is not monomorphic then it is not epimorphic nor isomorphic.

Recall G is monomorphic if $\text{null}(G) = \{0\}$.

Let's find null (G). take $u \in \text{null}(G)$. We check if u has to be 0. Write $u = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$.

$$G(u) = xu' - u = x(2ax+b) - (ax^2+bx+c)$$
$$= ax^2 - c$$

Since $u \in \text{null}(G)$, $G(u) = 0$ This means

$$ax^2 - c = 0 \quad \forall x \in \mathbb{R}$$

This implies $a = c = 0$ (for example by plugging $x=0$ and $x=1$).

We see that there is no constraint on b.

Thus,

$$\text{null}(G) = \{ u = bx : b \in \mathbb{R} \} = \text{span}\{x\}.$$

In particular, $\text{null}(G) \neq \{0\}$. Thus, G is not monomorphic.