MTH 342 Worksheet 2

Name: **Answer Key** **Recitation time:** __________

Show your work for each problem.

1. Let V be a vector space over a field F and let $\mathbf{0}$ be the zero vector in V. Below is a proof that $\alpha\mathbf{0} = \mathbf{0}$ for any scalar $\alpha \in F$. Fill in the blanks with the axiom used at each step:

proof. Let α be a scalar and let $-(\alpha\mathbf{0})$ be the additive inverse of $\alpha\mathbf{0}$. Then

$$\alpha\mathbf{0} = \alpha(\mathbf{0}+\mathbf{0}) \qquad \text{A3: Zero vector}$$

$$= \alpha\mathbf{0} + \alpha\mathbf{0} \qquad \text{D1: distribution over vector addition}$$

Now add $-(\alpha\mathbf{0})$ to both sides of the equation and simplify the left-hand side:

$$\alpha\mathbf{0} + [-(\alpha\mathbf{0})] = (\alpha\mathbf{0}+\alpha\mathbf{0}) + [-(\alpha\mathbf{0})]$$

$$\downarrow \qquad \text{A4: Additive inverse}$$

$$\mathbf{0} = (\alpha\mathbf{0}+\alpha\mathbf{0}) + [-(\alpha\mathbf{0})]$$

We need three steps to simplify the right-hand side and complete the proof:

$$\mathbf{0} = (\alpha\mathbf{0}+\alpha\mathbf{0}) + [-(\alpha\mathbf{0})]$$

$$\downarrow \qquad \text{A2: associativity of addition}$$

$$\mathbf{0} = \alpha\mathbf{0} + (\alpha\mathbf{0} + [-(\alpha\mathbf{0})])$$

$$\downarrow \qquad \text{A4: Additive inverse}$$

$$\mathbf{0} = \alpha\mathbf{0} + \mathbf{0}$$

$$\downarrow \qquad \text{A3: Zero vector}$$

$$\mathbf{0} = \alpha\mathbf{0}.$$

2. Let V be a vector space. For any $\mathbf{v} \in V$ let $-\mathbf{v}$ denote the additive inverse of $\mathbf{v}$. Prove that $-(-\mathbf{v}) = \mathbf{v}$ for any $\mathbf{v} \in V$.
(Hint: consider $\mathbf{v} + [-\mathbf{v}] + [-(-\mathbf{v})]$ and simplify in two different ways).

Solution: Let $\mathbf{v} \in V$. Then by associativity of addition we have

$$\Big(\mathbf{v} + [-\mathbf{v}]\Big) + [-(-\mathbf{v})] = \mathbf{v} + \Big([-\mathbf{v}] + [-(-\mathbf{v})]\Big).$$

The sums in each group of parentheses are sums of a vecor and its additive invers, so this simplifies to

$$\mathbf{0} + [-(-\mathbf{v})] = \mathbf{v} + \mathbf{0}.$$

Finally, we use the fact that $\mathbf{0}$ is the additive identity to conclude that

$$-(-\mathbf{v}) = \mathbf{v}.$$

3. Give an example of a non-empty subset S of $\mathbb{R}^2$ that is closed under scalar multiplication (for all $\mathbf{x} \in S$ and for all $c \in \mathbb{R}$, $c\mathbf{x} \in S$), but that is not a subspace of $\mathbb{R}^2$.

Solution: There are many possible answers. Remember that a subset S of a vector space is a subspace if it satisfies three axioms:

i) S is closed under addition.

ii) S is closed under scalar multiplication.

iii) S contains 0.

One example is

$$S = \{(a, 0) : a \in \mathbb{R}\} \cup \{(0, b) : b \in \mathbb{R}\}$$

This set is not closed under addition, since $(1, 0),\ (0, 1) \in S$ but

$$(1, 0) + (0, 1) = (1, 1) \notin S.$$

4. Let

$$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = x_1 - x_2\}.$$

a.) Prove that V is a vector space over $\mathbb{R}$.

Solution: Since $V \subseteq \mathbb{R}$, we only need to show that V is a subspace of $\mathbb{R}$. The three subspace axioms are

i) S contains the zero vector.

ii) S is closed under addition.

iii) S is closed under scalar multiplication.

Notice that any element of V can be written as $(x_1, x_2, x_1 - x_2)$ for $x_1, x_2 \in \mathbb{R}$.

Condition (i) is satisfied since $0 = 0 - 0$, so $(0, 0, 0) \in V$.

To prove (ii), let $(a, b, a - b)$ and $(c, d, c - d)$ be arbitrary elements of V. Then

$$\begin{aligned}(a, b, a - b) + (c, d, c - d) &= (a + c, b + d, a - b + c - d) \\ &= (a + c, b + d, (a + c) - (b + d))\end{aligned}$$

Since this sum satisfies the condition $x_3 = x_1 - x_2$, we have $(a, b, a - b) + (c, d, c - d) \in V$, so V is closed under addition.

To prove (iii), let $(a, b, a - b)$ be an arbitrary element of V and let $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}\lambda(a, b, a - b) &= (\lambda a, \lambda b, \lambda(a - b)) \\ &= (\lambda a, \lambda b, \lambda a - \lambda b)\end{aligned}$$

Since this element satisfies the condition $x_3 = x_1 - x_2$, we have $\lambda(a, b, a - b) \in V$, so V is closed under scalar multiplication.

b.) Find a basis for V (and prove that it is a basis). What is the dimension of V?

Solution: We can rewrite V as

$$\begin{aligned} V &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = x_1 - x_2\} \\ &= \{(x_1, x_2, x_1 - x_2) : x_1, x_2, x_3 \in \mathbb{R}\} \\ &= \{(x_1, 0, x_1) + (0, x_2, -x_2) : x_1, x_2, x_3 \in \mathbb{R}\} \\ &= \{x_1(1, 0, 1) + x_2(0, 1, -1) : x_1, x_2, x_3 \in \mathbb{R}\} \end{aligned}$$

So every element of V can be written as a linear combination of $\mathbf{v}_1 = (1, 0, 1)$ and $\mathbf{v}_2 = (0, 1, -1)$. Notice that $\mathbf{v}_1$ and $\mathbf{v}_2$ each satisfy the condition $x_3 = x_1 - x_2$, so $\mathbf{v}_1, \mathbf{v}_2 \in V$. These two facts together show that

$$V = \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2\}).$$

To show $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V, we must show that they are linearly independent. Let $c_1, c_2 \in \mathbb{R}$ and consider the equation

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= \mathbf{0} \\ &\downarrow \\ c_1(1, 0, 1) + c_2(0, 1, -1) &= (0, 0, 0) \\ &\downarrow \\ (c_1, c_2, c_1 - c_2) &= (0, 0, 0) \end{aligned}$$

In this last equation the first coordinate gives $c_1 = 0$ and the second coordinate gives $c_2 = 0$. Therefore B is linearly independent. Since B has exactly two elements, the dimension of V is 2.

5. The set $\mathbb{R}^{\mathbb{R}}$ of functions $f\colon \mathbb{R} \to \mathbb{R}$ with standard function addition and scalar multiplication forms a vector space over the field $F = \mathbb{R}$. Determine whether the following sets of functions in $\mathbb{R}^{\mathbb{R}}$ are linearly independent. If a set is linearly dependent, find a nontrivial linear combination equal to 0.

a.) $\{e^x, e^{x+2}\}$

Solution: Linearly dependent. Remember that

$$e^{x+2} = e^x e^2 = e^2 \cdot e^x.$$

Therefore

$$\begin{aligned} \boxed{-e^2(e^x) + (e^{x+2})} &= -e^2 \cdot e^x + e^2 \cdot e^x \\ &= 0 \end{aligned}$$

is a linear combination equal to 0 ($c_1 = -e^2$ and $c_2 = 1$).

b.) $\{\cos^2(x), \sin^2(x)\}$

Solution: Linearly independent. Let $c_1, c_2 \in \mathbb{R}$ be constants and consider the equation

$$c_1 \cos^2(x) + c_2 \sin^2(x) = 0$$

where this is true **for all** $x \in \mathbb{R}$. To prove that the functions are linearly independent, we must show that $c_1 = c_2 = 0$.

If $x = 0$ then the equation becomes

$$c_1(1) + c_2(0) = 0 \qquad \rightarrow \qquad c_1 = 0.$$

If $x = \pi/2$ then the equation becomes

$$c_1(0) + c_2(1) = 0 \qquad \rightarrow \qquad c_2 = 0.$$

c.) $\{\cos^2(x), \sin^2(x), 5\}$

Solution: Linearly dependent. Remember the trigonometric identity

$$\cos^2(x) + \sin^2(x) = 1.$$

Subtract 1 from both sides to get

$$\cos^2(x) + \sin^2(x) - 1 = 0.$$

Now we just need to write -1 as a $(-\frac{1}{5})5$:

$$\boxed{\cos^2(x) + \sin^2(x) + (-\tfrac{1}{5})5} = 0$$

($c_1 = 1$, $c_2 = 1$, and $c_3 = -\frac{1}{5}$).