MTH 342 Worksheet 3

Name: **Answer Key** **Recitation time:** ________

Show your work for each problem.

1. Determine (with proof) which of the following maps are linear.

a.) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

Solution: f is not linear. Notice that $f(1) = 1$ and $f(2) = 4$, but

$$f(1+2) = f(3) = 9 \neq 5 = f(1) + f(2).$$

b.) $A\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $A(x_1, x_2) = x_1 + 3x_2$.

Solution: A is linear. Let $(x_1, x_2), (z_1, z_2) \in \mathbb{R}^2$ and let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} A((x_1, x_1) + (z_1, z_2)) &= A(x_1 + z_1, x_2 + z_2) \\ &= (x_1 + z_1) + 3(x_2 + z_2) \\ &= (x_1 + 3x_2) + (z_1 + 3z_2) \\ &= A(x_1, x_2) + A(z_1, z_2), \end{aligned}$$

and

$$\begin{aligned} A(\alpha(x_1, x_1)) &= A(\alpha x_1, \alpha x_2) \\ &= \alpha x_1 + 3\alpha x_2 \\ &= \alpha(x_1 + 3x_2) \\ &= \alpha A(x_1, x_2), \end{aligned}$$

so A is linear.

2. Let $\mathbb{P}_2$ be the real vector space of polynomials of degree 2 or less. Let $S\colon \mathbb{P}_2 \to \mathbb{P}_2$ be given by

$$S(f(x)) = e^{-x}\frac{d}{dx}[e^x f(x)],$$

which is linear. Find the matrix representation $[S]_B$ where $B = \{1, x, x^2\}$.

Solution: To find the columns of $[S]_B$ we must apply S to each element of B and write the results in terms of the elements of B. The first element of B is 1, so calculate $S(1)$:

$$\begin{aligned} S(1) &= e^{-x}\frac{d}{dx}[e^x] \\ &= 1 \\ &= 1(1) + 0(x) + 0(x^2). \end{aligned}$$

The first column of $[S]_B$ is

$$[S(1)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The second element of B is x, so calculate $S(x)$:

$$\begin{aligned} S(x) &= e^{-x}\frac{d}{dx}[xe^x] \\ &= e^{-x}(e^x + xe^x) \\ &= 1 + x \\ &= 1(1) + 1(x) + 0(x^2) \end{aligned}$$

The second column of $[S]_B$ is

$$[S(x)]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The third element of B is x^2, so calculate $S(x^2)$:

$$\begin{aligned} S(x^2) &= e^{-x}\frac{d}{dx}[x^2 e^x] \\ &= e^{-x}(2xe^x + x^2 e^x) \\ &= 2x + x^2 \\ &= 0(1) + 2(x) + 1(x^2) \end{aligned}$$

The third column of $[S]_B$ is

$$[S(x^2)]_B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore

$$[S]_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Let V be a finite dimensional vector space. Then the following theorem holds:

Theorem (Theorem 2.6 in Linear Algebra Done Right)**.** The length of any linearly independent list of vectors in V is less than or equal to the length of any spanning list of vectors in V.

Use this theorem to prove that any two bases for V have the same number of elements.

Solution: Let B_1 and B_2 be bases for V. Then B_1 is linearly independent and B_2 is a spanning set for V, so $|B_1| \leq |B_2|$ (here $|S|$ denotes the size of a set S). Likewise, B_2 is linearly independent and B_1 is a spanning set for V, so $|B_2| \leq |B_1|$. Therefore $|B_1| = |B_2|$

4. Let V be a finite-dimensional vector space. Prove the following facts:

(a). Prove that any finite spanning set S for V contains a basis for V.
(Hint: Let $S = \{v_1, \dots, v_n\}$ and for each $j = 1, \dots, n$ remove v_j if $v_j \in \text{span}\{v_1, \dots, v_{j-1}\}$. Prove that the new set is a basis for V.)

Solution: If $V = \{0\}$ then the only basis for V is the empty set. Since the empty set is automatically a subset of S, we are done.

Now suppose $V \neq \{0\}$. Let $S = \{v_1, \dots, v_n\}$ be a spanning set for V. We will construct a basis B by removing elements from S. We will do this in steps.

Step 1: If $v_1 = 0$, then remove v_1. Otherwise, do not change the set.

Step j: If $v_j \in \text{span}\{v_1, \dots, v_{j-1}\}$, then remove v_j. Otherwise, do not change the set.

Let B be the set formed after applying n steps. We must prove that B is a basis for V.

- ***B* spans *V*:** Notice that at each step we only potentially remove an element if it is already in the span of the other elements. Therefore, each time we remove an element we do not change the span of the set. Since the original set S spans V, so does the new set B.
- ***B* is linearly independent:** We will use proof by contradiction. Suppose B is not linearly independent. Then there is a nontrivial linear combination in B that is equal to 0. We can write this as

$$c_1 v_1 + \cdots + c_n v_n = 0$$

where $c_i = 0$ whenever $v_i \notin B$. Let k be the greatest index such that $c_k \neq 0$ (therefore $v_k \in B$). Then we can solve the above equation for v_k. This means that $v_k \in \text{span}\{v_1, \dots, v_{k-1}\}$ (since $c_j = 0$ for $j > k$), but this contradicts our construction of B at step k.

(b). Suppose V has dimension n. Prove that any set of n vectors that spans V is a basis for V.

Solution. Let S be a set of n vectors that spans V. Then S can be reduced to a basis B for V by problem 4(a). However, since $\dim(V) = n$, B must have n elements, so we must not have removed any elements from S to form B. Therefore $S = B$, so S is a basis for V.

5. Let V be a vector space and suppose $v_1, \ldots, v_n \in V$ are linearly independent. Prove that the set $S = \{v_1, v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n\}$ is linearly independent.

Solution: Let $U = \operatorname{span}(v_1, \ldots, v_n)$, so $\dim(U) = n$. We want to show that S spans U, so that we can apply problem 4(b).

First note that no two elements of S (as given above) are equal. If two elements were equal, then we could rearrange the equation to get a nontrivial linear combination of the v_i to be equal to zero. This would contradict the assumption that $v_1, \ldots, v_n$ are linearly independent. Since no two vectors in S are equal, S has exactly n elements.

To see that S spans U, notice that we can write

$$v_2 = v_1 - (v_1 - v_2),$$

so $v_2 \in \operatorname{span}(S)$. Likewise, we can write

$$v_3 = v_2 - (v_2 - v_3).$$

Since $v_2 \in \operatorname{span}(S)$ and $(v_2 - v_3) \in S$, we also get $v_3 \in \operatorname{span}(S)$. Continuing in this way, we find $v_i \in \operatorname{span}(S)$ for all $i = 1, \ldots, n$. Therefore

$$U = \operatorname{span}(v_1, \ldots, v_n) \subseteq \operatorname{span}(S),$$

so S spans U. Since S has n vectors, problem 4(b) tells us that S is a basis for U. Since S is a basis, it must be linearly independent.