MTH 342 Worksheet 4

Name: **Answer Key** **Recitation time:** __________

Show your work for each problem.

1. Does there exist a linear map $f\colon \mathbb{R}^4 \to \mathbb{R}$ with null space

$$N = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = 3x_2,\ x_3 = x_4\}?$$

If yes, give an example of such a linear map. If no, give a proof.

Solution: No such linear map exists. Notice that we can rewrite N as

$$\begin{aligned} N &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = 3x_2,\ x_3 = x_4\} \\ &= \{(3x_2, x_2, x_3, x_3) : x_2, x_3 \in \mathbb{R}\} \\ &= \{x_2(3, 1, 0, 0) + x_3(0, 0, 1, 1) : x_2, x_3 \in \mathbb{R}\}. \end{aligned}$$

It is not difficult to show that N is a subspace of $\mathbb{R}^4$ with basis $\{(3, 1, 0, 0), (0, 0, 1, 1)\}$, so $\dim(N) = 2$.

Suppose N is the null space of $f\colon \mathbb{R}^4 \to \mathbb{R}$. Then

$$\text{nullity}(f) = \dim(N) = 2.$$

By the rank nullity theorem we must have

$$\begin{gathered} \text{rank}(f) + \text{nullity}(f) = \dim(\mathbb{R}^4) \\ \downarrow \\ \text{rank}(f) + 2 = 4 \\ \downarrow \\ \text{rank}(f) = 2 \end{gathered}$$

But this is impossible, because f maps to a 1-dimensional space, so $\text{rank}(f)$ is less than or equal to 1. Therefore our assumption that N is the null space of f was incorrect.

2. Let

$$\begin{aligned} U &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = x_3\} \\ V &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_3 = x_4,\ 2x_2 = x_4\}. \end{aligned}$$

a.) Find bases for U and V.

Solution: Rewrite U:

$$\begin{aligned} U &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = x_3\} \\ &= \{(x_1, x_1, x_1, x_4) : x_1, x_4 \in \mathbb{R}\} \\ &= \{x_1(1, 1, 1, 0) + x_4(0, 0, 0, 1) : x_1, x_4 \in \mathbb{R}\} \end{aligned}$$

so a basis for U is $\mathcal{B}_1 = \{(1, 1, 1, 0), (0, 0, 0, 1)\}$. Note that our rewriting of U shows that $U = \text{span}(\mathcal{B}_1)$, and it is not difficult to show that $\mathcal{B}_1$ is linearly independent. I have omitted the details.

Now rewrite V:

$$\begin{aligned} V &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_3 = x_4,\ 2x_2 = x_4\} \\ &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_3 = 2x_2 - x_1,\ 2x_2 = x_4\} \\ &= \{(x_1, x_2, 2x_2 - x_1, 2x_2) : x_1, x_2 \in \mathbb{R}\} \\ &= \{x_1(1, 0, -1, 0) + x_2(0, 1, 2, 2) : x_1, x_2 \in \mathbb{R}\} \end{aligned}$$

so a basis for V is $\mathcal{B}_2 = \{(1, 0, -1, 0), (0, 1, 2, 2)\}$.

b.) Find a basis for $U+V$. Is $U+V$ a direct sum of U and V?

Solution: Consider the set $\mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1$ and $\mathcal{B}_2$ are the bases for U and V as given in the solution to part (a). Form a matrix A with the elements of $\mathcal{B}_1 \cup \mathcal{B}_2$ as rows:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

Now find the reduced row echelon form of A:

$$A \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rows of this reduced matrix represent a basis for $U+V$:

$$\mathcal{B} = \{(1,1,1,0), (0,0,0,1), (1,0,-1,0)\}.$$

Then $\dim(U+V) = 3$ and from part (a) we can see that $\dim(U) = \dim(V) = 2$. Therefore

$$\dim(U+V) = 3 \neq 4 = \dim(U) + \dim(V),$$

so $U+V$ is not a direct sum of U and V.

3. Let $V = \mathbb{R}^4$ and

$$W = \{a + bx + cx^2 : a, b, c \in \mathbb{R}, 2a + 2b - c = 0\}$$

viewed as real vector spaces.

a.) Construct a linear map $f\colon V \to W$ with rank 2.

Solution: There are many possible answers here.

Let $(a, b, c, d) \in \mathbb{R}^4$. The easiest way to ensure f has rank **less than or equal to** 2 is to only use 2 components (say a and b) in the output. For example, we could choose

$$f(a, b, c, d) = a + bx + (2a + 2b)x^2.$$

Notice that the $(2a + 2b)x^2$ term is included so that the image of f is contained in W. We will see by part (b) that $\dim(\text{range}(f)) = 2$, so $\text{rank}(f) = 2$.

b.) Find a basis for $\text{range}(f)$.

Solution: This answer depends on your answer to part (a). In our case,

$$\begin{aligned}\text{range}(f) &= \{f(a, b, c, d) : a, b, c, d \in \mathbb{R}\}\\ &= \{a + bx + (2a + 2b)x^2 : a, b \in \mathbb{R}\}\\ &= \{a(1 + 2x^2) + b(x + 2x^2) : a, b \in \mathbb{R}\} \qquad (*).\end{aligned}$$

We propose $B_1 = \{1 + 2x^2, x + 2x^2\}$ as a basis for $\text{range}(f)$. Notice that

$$\begin{aligned}f(1, 0, 0, 0) &= 1 + 2x^2\\ f(0, 1, 0, 0) &= x + 2x^2\end{aligned}$$

so $B_1 \subset \text{range}(f)$. We can see from $(*)$ that $\text{range}(f) = \text{span}(B_1)$.

Finally, we need to check that B_1 is linearly independent. Let $c_1, c_2 \in \mathbb{R}$ and suppose

$$c_1(1 + 2x^2) + c_2(x + 2x^2) = 0.$$

Setting $x = 0$ gives $c_1 = 0$, so the above equation becomes

$$c_2(x + 2x^2) = 0.$$

Setting $x = 1$ gives $3c_2 = 0$, so $c_2 = 0$. Therefore the functions are linearly independent.

c.) What is the nullity of f?

Solution: This answer does **not** depend on your answer to part (a).

By the rank-nullity theorem

$$\begin{aligned}\text{rank}(f) + \text{nullity}(f) &= \dim(V)\\ &\downarrow\\ 2 + \text{nullity}(f) &= 4\\ &\downarrow\\ \text{nullity}(f) &= 2\end{aligned}$$

d.) Find a basis for null(f).

Solution: This answer depends on your answer to part (a).

We want to find all $(a, b, c, d) \in \mathbb{R}$ such that

$$\begin{aligned} 0 &= f(a, b, c, d) \\ &= a + bx + (2a + 2b)x^2 \\ &= a(1) + b(x) + (2a + 2b)x^2. \end{aligned}$$

Since the set $\{1, x, x^2\}$ is linearly independent, the only way for this to be true is if $a = b = 0$. Therefore $(a, b, c, d) \in \text{null}(f)$ if and only if $a = b = 0$. That is,

$$\text{null}(f) = \{(0, 0, c, d) : c, d \in \mathbb{R}\}.$$

It is easy to check that $B_2 = \{(0, 0, 1, 0), (0, 0, 0, 1)\}$ is a basis for null(f).

4. Find a vector space V with subspaces U_1, U_2, and W such that

a.) $U_1 + W = U_2 + W$, but $U_1 \neq U_2$.

Solution: Let

$$\begin{aligned} V &= \mathbb{R} \\ U_1 &= \mathbb{R} \\ U_2 &= \{0\} \\ W &= \mathbb{R}. \end{aligned}$$

Then

$$U_1 + W = \mathbb{R} = U_2 + W,$$

but

$$U_1 = \mathbb{R} \neq \{0\} = U_2.$$

b.) $U_1 \oplus W = U_2 \oplus W$, but $U_1 \neq U_2$.

Solution: Notice that a valid solution to part (b) is also a solution to part (a).

Let

$$\begin{aligned} V &= \mathbb{R}^2 \\ U_1 &= \{(x, 0) : x \in \mathbb{R}\} \\ U_2 &= \{(0, y) : y \in \mathbb{R}\} \\ W &= \{(z, z) : z \in \mathbb{R}\}. \end{aligned}$$

It is not hard to check that

$$U_1 + W = \text{span}\{(1, 0), (1, 1)\} = \mathbb{R}^2$$

and

$$U_2 + W = \text{span}\{(0, 1), (1, 1)\} = \mathbb{R}^2$$

so $U_1 + W = U_2 + W$.

Now notice that if $(a, b) \in U_1 \cap W$ then $b = 0$ (since $(a, b) \in U_1$) and $a = b = 0$ (since $(a, b) \in W$). Therefore $U_1 \cap W = \{0\}$, so $U_1 + W$ is a direct sum of U_1 and W. A similar argument shows that $U_2 + W$ is a direct sum of U_2 and W.

Finally, notice that $U_1 \neq U_2$, since for example $(1, 0) \in U_1$ but $(1, 0) \notin U_2$.