MTH 342 Worksheet 5

Name: **Answer Key** **Recitation time:** __________

Show your work for each problem.

1. Consider the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$. Use a change-of-basis matrix to find the coordinates of $(1\ 2\ 3)^T$ with respect to the basis B.

Solution: Let E be the standard basis for $\mathbb{R}^3$. We want to find a matrix that converts vectors in standard coordinates (i.e., coordinates with respect to E) into vectors in B coordinates. It is easy to go the opposite direction. The matrix that converts B coordinates to standard coordinates is

$$[\mathrm{id}]_{E,B} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

To get the matrix that converts E coordinates to B coordinates, simply find the inverse:

$$\begin{aligned} [\mathrm{id}]_{B,E} &= [\mathrm{id}]_{E,B}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 3 & -2 & -1 \\ 1 & -1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \end{aligned}$$

We can now find the coordinates of $(1, 2, 3)^T$ with respect to B using matrix multiplication:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_B = \begin{bmatrix} 3 & -2 & -1 \\ 1 & -1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 1 \end{bmatrix}.$$

This means that

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -4 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

which can easily be verified.

2. Let V, W be vector spaces over a field F and let $f\colon V \to W$ and $g\colon W \to V$ be linear maps such that

$$g \circ f(\mathbf{v}) = \mathbf{v} \qquad \text{for all } \mathbf{v} \in V$$

(a) Prove that f is a monomorphism.

Solution: We want to show that f is injective. That is, we want to show that if $f(v_1) = f(v_2)$ for some $v_1, v_2 \in V$, then $v_1 = v_2$.

Let $v_1, v_2 \in V$ and suppose that $f(v_1) = f(v_2)$. Applying g to both sides of this equation gives

$$\begin{gathered} f(v_1) = f(v_2) \\ \downarrow \\ g \circ f(v_1) = g \circ f(v_2) \\ \downarrow \\ v_1 = v_2, \end{gathered}$$

so f is injective and hence a monomorphism.

(b) Prove that g is an epimorphism.

Solution: Let v be an arbitrary element of V. To prove that g is an epimorphism, we must show that there exists some $w \in W$ such that $g(w) = v$.

Let $w = f(v)$.

$$\begin{aligned} g(w) = &= g(f(v)) \\ &= g \circ f(v) \\ &= v, \end{aligned}$$

so g is an epimorphism.

(c) What can we conclude about the relationship between $\dim(V)$ and $\dim(W)$?

Solution: We can conclude that $\dim(W) \geq \dim(V)$.

We will use a proof by contradiction. In order to reach a contradiction, assume that $\dim(W) < \dim(V)$. The rank-nullity theorem tells us

$$\begin{aligned} \operatorname{rank}(f) + \operatorname{nullity}(f) &= \dim(V) \\ &\downarrow \\ \operatorname{nullity}(f) &= \dim(V) - \operatorname{rank}(f) \\ &\geq \dim(V) - \dim(W) && \text{since } \operatorname{rank}(f) \leq \dim(W) \\ &> 0 && \text{since } \dim(W) < \dim(V) \text{ by assumption} \end{aligned}$$

Therefore f is not injective (i.e., not a monomorphism), contradicting part (a).

A similar proof can be constructed using the rank-nullity theorem applied to g.

3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix}.$$

The eigenvalues of A are $\lambda = 1, 2, 4$. Let

$$E_1 = \{v \in \mathbb{R}^3 : Av = v\}$$
$$E_2 = \{v \in \mathbb{R}^3 : Av = 2v\}$$
$$E_4 = \{v \in \mathbb{R}^3 : Av = 4v\}$$

(a) Find bases for E_1, E_2, and E_4.

Solution: Given any $\lambda \in \mathbb{R}$, the equation $Av = \lambda v$ can be rewritten as

$$\begin{gathered} Av = \lambda v \\ \downarrow \\ Av - \lambda v = 0 \\ \downarrow \\ (A - \lambda I)v = 0 \end{gathered}$$

Where I is the identity matrix.

To find E_1, we solve the equation $(A - I)v = 0$ for $v = (v_1, v_2, v_3)$. That is,

$$\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Write this is equation in augmented form and use row reduction:

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right]$$
$$\xrightarrow{R_3 + R_1 \to R_3} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$
$$\xrightarrow{\frac{1}{3}R_3 \to R_3} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore $v_2 = v_3 = 0$ and v_1 is a free variable, so

$$E_1 = \left\{ \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} : v_1 \in \mathbb{R} \right\}.$$

Then E_1 has basis

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

For E_2 and E_4 I will not write all of the row reduction details.

The augmented matrix for E_2 is

$$\left[\begin{array}{ccc|c} 1-2 & 1 & 0 & 0 \\ 0 & 2-2 & 0 & 0 \\ 0 & -1 & 4-2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore $v_1 = 2v_3$ and $v_2 = 2v_3$, where v_3 is a free variable. Thus

$$E_2 = \left\{ \begin{bmatrix} 2v_3 \\ 2v_3 \\ v_3 \end{bmatrix} : v_3 \in \mathbb{R} \right\}$$

has basis

$$B_2 = \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

The augmented matrix for E_4 is

$$\left[\begin{array}{ccc|c} 1-4 & 1 & 0 & 0 \\ 0 & 2-4 & 0 & 0 \\ 0 & -1 & 4-4 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore $v_1 = v_2 = 0$ and v_3 is a free variable, so

$$E_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix} : v_3 \in \mathbb{R} \right\}$$

has basis

$$B_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) Is the sum $E_1 + E_2 + E_4$ a direct sum?

Solution: Yes. Let

$$B = B_1 \cup B_2 \cup B_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

so $E_1 + E_2 + E_4 = \operatorname{span}(B)$. Then

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and thus $E_1 + E_2 + E_4$ contains the standard basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$. Therefore $E_1 + E_2 + E_4 = \mathbb{R}^3$. Calculating dimensions gives

$$\begin{aligned} \dim(E_1) + \dim(E_2) + \dim(E_4) &= 1 + 1 + 1 \\ &= 3 \\ &= \dim(\mathbb{R}^3) \end{aligned}$$

so $E_1 + E_2 + E_4$ is a direct sum of E_1, E_2, and E_4, and can be written as $E_1 \oplus E_2 \oplus E_4$.

4. Let

$$B = \{2 + 2x,\ 3 + x\} \qquad \text{and} \qquad C = \{1,\ 1 + x\}$$

which are bases for $P_1(\mathbb{R})$. Find the change of basis matrix that converts coordinates with respect to B into coordinates with respect to C (i.e., find $[\text{id}]_{C,B}$).

Solution: Let $E = \{1, x\}$ which is a basis for $P_1(\mathbb{R})$. We will use the following fact:

$$[\text{id}]_{C,B} = [\text{id}]_{C,E}[\text{id}]_{E,B}.$$

We now need to calculate these two matrices. Note that writing the elements of C in E coordinates gives

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\},$$

so

$$\begin{aligned} [\text{id}]_{C,E} &= [\text{id}]_{E,C}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Likewise, the elements of B written in E coordinates are

$$\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\},$$

so

$$[\text{id}]_{E,B} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}.$$

Therefore

$$\begin{aligned} [\text{id}]_{C,B} &= [\text{id}]_{C,E}[\text{id}]_{E,B} \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}. \end{aligned}$$