MTH 342 Worksheet 6

Name: Answer Key **Recitation time:** ______

Show your work for each problem.

1. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $f(v) = Av$ where

$$A = \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix}.$$

Find all eigenvalues and eigenvectors of A. Is f diagonalizable? If it is, express $V = \mathbb{R}^3$ as a direct sum of one-dimensional invariant subspaces under f; then find a basis of V in which f is represented by a diagonal matrix.

Solution: First find the characteristic polynomial of A:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 4-\lambda & -3 & -3 \\ 3 & -2-\lambda & -3 \\ -1 & 1 & 2-\lambda \end{vmatrix} \\
&= (4-\lambda)\begin{vmatrix} -2-\lambda & -3 \\ 1 & 2-\lambda \end{vmatrix} - 3\begin{vmatrix} -3 & -3 \\ 1 & 2-\lambda \end{vmatrix} + (-1)\begin{vmatrix} -3 & -3 \\ -2-\lambda & -3 \end{vmatrix} \\
&= (4-\lambda)[(-2-\lambda)(2-\lambda) - (-3)(1)] \\
&\quad - 3[(-3)(2-\lambda) - (-3)(1)] \\
&\quad - [(-3)(-3) - (-3)(-2-\lambda)] \\
&= (4-\lambda)(\lambda^2 - 1) - 3(3\lambda - 3) - (-3\lambda + 3) \\
&= (4-\lambda)(\lambda-1)(\lambda+1) - 3(3)(\lambda-1) - (-3)(\lambda-1) \\
&= (\lambda-1)[(4-\lambda)(\lambda+1) - 9 + 3] \\
&= (\lambda-1)[-\lambda^2 + 3\lambda - 2] \\
&= (\lambda-1)[-(\lambda-1)(\lambda-2)] \\
&= -(\lambda-1)^2(\lambda-2)
\end{aligned}$$

Setting $-(\lambda-1)^2(\lambda-2) = 0$ gives eigenvalues $\lambda = 1$ and $\lambda = 2$.

We still do not know if f is diagonalizable. We now need to find the eigenvectors of A. This means solving the equation

$$(A - \lambda I)v = 0.$$

where $v = (v_1, v_2, v_3)$.

First let $\lambda = -1$. The augmented form of the above equation is

$$\left[\begin{array}{ccc|c} 4-1 & -3 & -3 & 0 \\ 3 & -2-1 & -3 & 0 \\ -1 & 1 & 2-1 & 0 \end{array}\right] \quad \to \quad \left[\begin{array}{ccc|c} 3 & -3 & -3 & 0 \\ 3 & -3 & -3 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right]$$

Row reduction gives

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $v_1 - v_2 - v_3 = 0$. Adding $v_2 + v_3$ to both sides gives $v_1 = v_2 + v_3$, so the eigenvectors v can be written as

$$v = \begin{bmatrix} v_2 + v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(continued on next page)

Therefore a basis for the eigenspace E_1 is

$$B_1 = \{(1,1,0),(1,0,1)\}$$

Now let $\lambda = 2$. We follow the same process of solving $(A - \lambda I)v = 0$:

$$\left[\begin{array}{ccc|c} 4-2 & -3 & -3 & 0 \\ 3 & -2-2 & -3 & 0 \\ -1 & 1 & 2-2 & 0 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{ccc|c} 2 & -3 & -3 & 0 \\ 3 & -4 & -3 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right]$$

Row reduction gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right]$$

so we get the system of equations

$$v_1 + 3v_3 = 0$$
$$v_2 + 3v_3 = 0.$$

This means that $v_1 = v_2 = -3v_3$, so the eigenvector v can be written as

$$v = \begin{bmatrix} -3v_3 \\ -3v_3 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}.$$

A basis for E_2 is

$$B_2 = \{(-3,-3,1)\}.$$

Since $\dim(E_1) + \dim(E_2) = 2 + 1 = 3 = \dim(V)$ the linear map f is diagonalizable. Under the basis

$$B = B_1 \cup B_2 = \{(1,1,0),(1,0,1),(-3,-3,1)\} = \{w_1, w_2, w_3\}$$

the map f is represented by the diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

and

$$V = \mathbb{R}^3 = \mathbb{R}w_1 \oplus \mathbb{R}w_2 \oplus \mathbb{R}w_3$$

where each $\mathbb{R}w_i$ is an invariant subspace under f.

2. Consider the real vector space $\mathbb{R}^{\mathbb{N}}$ and let $F\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the left-shift operator

$$F(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots),$$

which is linear. Find all eigenvalues and eigenvectors of F.

Solution: Here we are working with an infinite dimensional vector space, so we cannot simply find the eigenvectors of a matrix. We must use the definition of the eigenvector.

The eigenvectors of F are the vectors $v = (x_1, x_2 \dots) \in \mathbb{R}^{\mathbb{N}}$ such that $F(v) = \lambda v$ for some $\lambda \in \mathbb{R}$. In this case we have

$$\begin{aligned}(x_2, x_3, x_4, \dots) &= \lambda(x_1, x_2, x_3, \dots)\\ &= (\lambda x_1, \lambda x_2, \lambda x_3, \dots)\end{aligned}$$

so

$$\begin{aligned}x_2 &= \lambda x_1\\ x_3 &= \lambda x_2\\ x_4 &= \lambda x_3\\ &\vdots\end{aligned}$$

We can now write every coordinate in terms of x_1:

$$\begin{aligned}x_2 &= \lambda x_1\\ x_3 &= \lambda x_2 = \lambda(\lambda x_1) = \lambda^2 x_1\\ x_4 &= \lambda x_3 = \lambda(\lambda^2 x_1) = \lambda^3 x_1\\ &\vdots\end{aligned}$$

so the eigenvector v looks like

$$\begin{aligned}v &= (x_1, \lambda x_1, \lambda^2 x_1, \lambda^3 x_1, \dots)\\ &= x_1(1, \lambda, \lambda^2, \lambda^3, \dots).\end{aligned}$$

There is no restriction on λ in our work above, so every element $\lambda \in \mathbb{R}$ is an eigenvalue. The eigenspace associated to the eigenvalue λ is

$$E_\lambda = \{x_1(1, \lambda, \lambda^2, \lambda^3, \dots) : x_1 \in \mathbb{R}\}$$

which has basis

$$B_\lambda = \{(1, \lambda, \lambda^2, \lambda^3, \dots)\}.$$

3. Let $g\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the linear map given by $g(v) = Bv$ where

$$B = \begin{bmatrix} i & 1 \\ 2 & 1+i \end{bmatrix}.$$

(a) Find the eigenvalues of g. Why does this show that g is diagonalizable?

Solution: Calculate the characteristic polynomial:

$$\begin{aligned} \det(B - \lambda I) &= \begin{vmatrix} i-\lambda & 1 \\ 2 & 1+i-\lambda \end{vmatrix} \\ &= (i-\lambda)(1+i-\lambda) - 2 \\ &= \lambda^2 - (1+2i)\lambda + (-3+i) \end{aligned}$$

To solve $\det(A - \lambda I) = 0$ we can use the quadratic formula:

$$\begin{aligned} \lambda &= \frac{(1+2i) \pm \sqrt{[-(1+2i)]^2 - 4(-3+i)}}{2} \\ &= \frac{(1+2i) \pm \sqrt{-3+4i+12-4i)}}{2} \\ &= \frac{(1+2i) \pm \sqrt{9}}{2} \\ &= \frac{(1+2i) \pm 3}{2} \end{aligned}$$

so

$$\lambda = 2+i \qquad \text{or} \qquad \lambda = -1+i.$$

Since $\mathbb{C}^2$ is 2-dimensional and g has two distinct roots, this means that g is diagonalizable.

(b) Find a basis of $\mathbb{C}^2$ in which g is represented by a diagonal matrix.

Solution: To find the eigenvectors, solve $(A - \lambda I)v = 0$ where $v = (v_1, v_2) \in \mathbb{C}^2$. First let $\lambda = 2 + i$:

$$\left[\begin{array}{cc|c} i-(2+i) & 1 & 0 \\ 2 & 1+i-(2+i) & 0 \end{array}\right] \xrightarrow{\text{simplify}} \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_1 \to R_2} \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The first line of the reduced matrix gives $-2v_1 + v_2 = 0$, so $v_2 = 2v_1$. We can write the eigenvector v as

$$\begin{bmatrix} v_1 \\ 2v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

so a basis for E_{2+i} is

$$B_{2+i} = \{(1,2)\}$$

Now let $\lambda = -1 + i$:

$$\left[\begin{array}{cc|c} i-(-1+i) & 1 & 0 \\ 2 & 1+i-(-1+i) & 0 \end{array}\right] \xrightarrow{\text{simplify}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2-2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Then $v_1 + v_2 = 0$, so $v_2 = -v_1$. We can write the eigenvector v as

$$\begin{bmatrix} v_1 \\ -v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

so a basis for E_{-1+i} is

$$B_{-1+i} = \{(1,-1)\}$$

Let

$$B = \{(1,2),(1,-1)\} = \{w_1, w_2\}.$$

Then B is

Therefore $\mathbb{C}^2$ can be written as

4. Let V be a real vector space. Suppose $T\colon V \to V$ is linear and satisfies $T^2 - T = 2I_V$, $T \neq -I_V$, and $T \neq 2I_V$. Prove that the set of eigenvalues of T is $\{-1, 2\}$.

Solution: First, it is helpful to rearrange the equation

$$\begin{gathered} T^2 - T = 2I_V \\ \downarrow \\ T^2 - T - 2I_V = 0. \end{gathered}$$

Suppose that λ is an eigenvalue of T, and let v be an eigenvector of T associated to λ. Consider the equation of linear maps $T^2 - T = 2I_V$. If we apply the map on each side of the equation to the vector v, we get

$$\begin{aligned} (T^2 - T - 2I_V)v &= 0 \\ &\downarrow \\ T^2 v - Tv - 2I_V v &= 0 \\ &\downarrow \\ \lambda^2 v - \lambda v - 2v &= 0. \\ &\downarrow \\ (\lambda^2 - \lambda - 2)v &= 0. \end{aligned}$$

Since v is an eigenvector, it is nonzero, so the only way to make the above equation true is if

$$\lambda^2 - \lambda - 2 = 0.$$

Factoring shows that $(\lambda + 1)(\lambda - 2) = 0$, so the only possible values for λ are -1 and 2.

We now want to show that both -1 and 2 occur as eigenvalues. To do this, it is helpful to factor $T^2 - T - 2I_V$. We can factor this just as we would factor the real polynomial $x^2 - x - 2$:

$$T^2 - T - 2I_V = (T + I_V)(T - 2I_V),$$

To see that the factorization works, you can apply the operator on each side of the above equation to a vector u and carefully use the rules of operator addition to confirm that you get the same result for each side. The equation then becomes

$$(T + I_V)(T - 2I_V) = 0$$

Since $T \neq 2I_V$ there is a vector u such that $w = (T - 2I_V)u \neq 0$. Then we must have

$$(T + I_V)(T - 2I_V)u = 0 \qquad \to \qquad (T + I_V)w = 0,$$

so we get

$$Tw + w = 0 \qquad \to \qquad Tw = (-1)w.$$

Therefore w is an eigenvalue of T with eigenvalue $\lambda = -1$ (note that w is nonzero, so it is indeed an eigenvector).

To show that 2 is an eigenvalue of T we would want to factor $T^2 - T - 2I_V$ as

$$T^2 - T - 2I_V = (T - 2I_V)(T + I_V).$$

which is also a valid factorization. We can then use and argument very similar to the one above to prove that 2 is an eigenvalue of T. I will omit the details.