MTH 342 Worksheet 7

Name: Answer Key **Recitation time:** ________

Show your work for each problem.

Let V be a vector space over a field F. Recall that an *inner product* is an operator $(\cdot,\cdot) : V \times V \to F$ satisfying the following properties:

- **Positivity:** $(v, v) \geq 0$ for all $v \in V$.
- **Definiteness:** $(v, v) = 0$ if and only if $v = 0$.
- **Additivity in the first component:** $(u + v, w) = (u, w) + (v, w)$ for all $u, v, w \in V$.
- **Homogeneity in the first component:** $(\lambda u, v) = \lambda(u, v)$ for all $\lambda \in F$ and $u, v \in V$.
- **Conjugate symmetry:** $(u, v) = \overline{(v, u)}$ for all $u, v \in V$.

1. On $\mathbb{R}^3$ define the operator

$$(x, y) = x_1y_1 + x_3y_3$$

where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

(a) Prove that this is not an inner product on $\mathbb{R}^3$.

Solution: Let

$$x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then

$$(x, x) = 0,$$

but $x \neq 0$, so this operator is not definite, and hence not an inner product.

(b) Which properties of an inner product does this operator satisfy?

Solution: All other properties (besides definiteness) are satisfied. Let u, v, w be arbitrary elements of $\mathbb{R}^3$ written as

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \qquad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

and let $\lambda \in F = \mathbb{R}$.

- **Positivity:**

$$(v, v) = (v_1)^2 + (v_3)^2 \geq 0 + 0 \geq 0.$$

- **Additivity in the first component:**

$$\begin{aligned} (u + v, w) &= (u_1 + v_1)w_1 + (u_3 + v_3)w_3 \\ &= u_1w_1 + v_1w_1 + u_3w_3 + v_3w_3 \\ &= (u_1w_1 + u_3w_3) + (v_1w_1 + v_3w_3) \\ &= (u, w) + (v, w). \end{aligned}$$

- **Homogeneity in the first component:**

$$\begin{aligned}(\lambda u, v) &= (\lambda u_1)v_1 + (\lambda u_3)v_3 \\ &= \lambda(u_1v_1 + u_3v_3) \\ &= \lambda(u, v)\end{aligned}$$

- **Conjugate symmetry:** Note that for any real number a, the complex conjugate is $\overline{a} = a$. Now

$$\begin{aligned}(u, v) &= u_1v_1 + u_3v_3 \\ &= v_1u_1 + v_3u_3 \\ &= \overline{v_1u_1 + v_3u_3} \qquad \text{since } v_1u_1 + v_3u_3 \text{ is a real number} \\ &= \overline{(v, u)}.\end{aligned}$$

2. On $\mathbb{C}^2$ let $u_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $u_2 = \begin{bmatrix}0\\i\end{bmatrix}$. Suppose $(\cdot,\cdot)$ is an inner product on $\mathbb{C}^2$ that satisfies

$$(u_1, u_1) = 1, \qquad (u_1, u_2) = -i, \qquad (u_2, u_2) = 2.$$

Compute $\left(\begin{bmatrix}1\\i\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right)$.
(**Hint:** you will need to use *conjugate* linearity in the second component.)

Solution: We start by writing the each component in terms of u_1 and u_2:

$$\begin{bmatrix}1\\i\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\i\end{bmatrix} = u_1 + u_2$$

$$\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\i\end{bmatrix} = u_1 + iu_2$$

We can now calculate

$$\begin{aligned}\left(\begin{bmatrix}1\\i\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right) &= (u_1 + u_2, u_1 + iu_2) \\ &= (u_1, u_1 + iu_2) + (u_2, u_1 + iu_2) \qquad \text{(linearity in the first component)} \\ &= (u_1, u_1) + \bar{i}(u_1, u_2) + (u_2, u_1) + \bar{i}(u_2, u_2) \quad \left(\begin{array}{l}\text{conjugate linearity in} \\ \text{the second component}\end{array}\right) \\ &= (u_1, u_1) + \bar{i}(u_1, u_2) + \overline{(u_1, u_2)} + \bar{i}(u_2, u_2) \quad \text{(conjugate symmetry)} \\ &= (u_1, u_1) - i(u_1, u_2) + \overline{(u_1, u_2)} - i(u_2, u_2) \\ &= 1 - i(-i) + i - i(2) \\ &= 1 - 1 + i - 2i \\ &= -i.\end{aligned}$$

3. Let V be an inner product space with inner product $(\cdot,\cdot)$, and let $v_1, v_2 \in V$. Prove that if

$$(v_1, w) = (v_2, w)$$

for all $w \in V$, then $v_1 = v_2$.

Solution: Rewrite the equation as

$$(v_1, w) - (v_2, w) = 0.$$

Now we can use the definition of an inner product (additivity in the first component) to get

$$\begin{aligned}(v_1, w) - (v_2, w) &= 0\\ &\downarrow\\ (v_1 - v_2, w) &= 0\end{aligned}$$

for all $w \in V$. If we let $w = v_1 - v_2$, then this becomes

$$(v_1 - v_2, v_1 - v_2) = 0.$$

By the definition of an inner product (definiteness), we get

$$v_1 - v_2 = 0.$$

Adding v_2 to both sides gives $v_1 = v_2$.

4. Let a and b be integers and suppose that

$$\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = x_1y_1 + ax_1y_2 + bx_2y_1 + x_2y_2$$

defines an inner product on $\mathbb{R}^2$.

(a) Prove that $a = b$. (**Hint:** Find vectors $u, v \in \mathbb{R}^2$ such that $(u, v) = a$ and $(v, u) = b$. Why does this show that $a = b$?)

Solution: Following the hint, let

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then $(u, v) = a$ and $(v, u) = b$. Now we have by conjugate symmetry that

$$\begin{aligned} a &= (u, v)\\ &= \overline{(v, u)}.\end{aligned}$$

Since this is a real inner product space, the inner product of two vectors is a real number. Since all real numbers are equal to their complex conjugate, we get

$$\begin{aligned} a &= (u, v)\\ &= \overline{(v, u)}\\ &= (v, u)\\ &= b.\end{aligned}$$

(b) What are the possible values of a and b?

Hint: calculate $\left(\begin{bmatrix} -a \\ 1 \end{bmatrix}, \begin{bmatrix} -a \\ 1 \end{bmatrix} \right)$.

Solution: We must have $a = 0$ and $b = 0$. From the hint,

$$\begin{aligned} \left(\begin{bmatrix} -a \\ 1 \end{bmatrix}, \begin{bmatrix} -a \\ 1 \end{bmatrix} \right) &= (-a)^2 + a(-a) + b(-a) + 1 \\ &= a^2 - a^2 - ab + 1 \\ &= -ab + 1 \\ &= -a^2 + 1 \qquad \text{since } a = b \text{ by part (a).} \end{aligned}$$

We want this to be an inner product, so we need to have $(v, v) > 0$ for any nonzero $v \in \mathbb{R}^2$ (definiteness). Therefore

$$-a^2 + 1 > 0.$$

Adding a^2 to both sides gives $1 > a^2$, and the only integer that satisfies this inequality is $a = 0$. Therefore $a = b = 0$.

Proving that this is indeed an inner product is very similar to a homework problem (HW 5, problem 2). I will leave this as an exercise.