# MTH 342 Worksheet 8

**Name:** **Answer Key** **Recitation time:** ________

Show your work for each problem.

**1.** Let $V$ be a real inner product space with inner product $(\cdot, \cdot)$. Prove that

$$(v + w, v - w) = \|v\|^2 - \|w\|^2$$

for all $v, w \in V$. Conclude that $v + w$ and $v - w$ are orthogonal if and only if the norms of $v$ and $w$ are equal.
(Note: in $V = \mathbb{R}^2$ this can be used to prove that the diagonals of a rhombus are perpendicular).

**Solution:** Let $v, w \in V$. We simply need to expand $(v + w, v - w)$ using the rules of an inner product space:

$$\begin{aligned}
(v + w, v - w) &= (v, v - w) + (w, v - w) \\
&= (v, v) - (v, w) + (w, v) - (w, w) \\
&= \|v\|^2 - (v, w) + (w, v) - \|w\|^2 \\
&= \|v\|^2 - (v, w) + \overline{(v, w)} - \|w\|^2 \\
&= \|v\|^2 - (v, w) + (v, w) - \|w\|^2 \qquad \text{(since } V \text{ is a real inner product space)} \\
&= \|v\|^2 - \|w\|^2.
\end{aligned}$$

Now we will prove the if and only if statement:

- $(\Rightarrow)$ Suppose $v + w$ and $v - w$ are orthogonal. Then

$$\|v\|^2 - \|w\|^2 = (v + w, v - w) = 0,$$

  so $\|v\|^2 = \|w\|^2$, and hence $\|v\| = \|w\|$. Note that we do not need to worry about $\|v\|$ and $\|w\|$ having different signs, because $\|v\|$ and $\|w\|$ must be nonnegative by the positivity of the norm.

- $(\Leftarrow)$ Suppose $\|v\| = \|w\|$ then we have

$$(v + w, v - w) = \|v\|^2 - \|w\|^2 = 0,$$

  so $v + w$ and $v - w$ are orthogonal.

**2.** Prove that

$$\|(x, y)\| = \max\{|x|, |y|\}$$

defines a norm on $\mathbb{R}^2$.

**Solution:** We need to prove each property of a norm. Let $(x, y), (a, b) \in \mathbb{R}^2$ and let $\lambda \in \mathbb{R}$.

- **Positivity:** $\|(x, y)\|$ is always the absolute value of a real number, so $\|(x, y)\| \geq 0$.
- **Definiteness:** This is an if and only if statement, so we need to prove both directions:
  - $\|(0, 0)\| = \max\{|0|, |0|\} = \max\{0, 0\} = 0$.
  - Suppose $\|(x, y)\| = 0$. Then the largest absolute value of either component is 0. Since the absolute value of any number is at least 0, the only way for this to occur is if $|x| = 0$ and $|y| = 0$. By the definiteness of the absolute value we get $x = 0$ and $y = 0$.

- **Homogeneity:** First suppose $|x| \geq |y|$. Multiplying both sides of the inequality by $|\lambda|$ gives

$$|\lambda||x| \geq |\lambda||y|$$

(note that the inequality does not change directions, because $|\lambda|$ is nonnegative). Now

$$\begin{aligned}\|\lambda(x,y)\| &= \|(\lambda x, \lambda y)\| \\ &= \max\{|\lambda x|, |\lambda y|\} \\ &= \max\{|\lambda||x|, |\lambda||y|\} \\ &= |\lambda||x| \\ &= |\lambda| \max\{|x|, |y|\} \\ &= |\lambda|\|(x,y)\|.\end{aligned}$$

The proof in the case $|x| \leq |y|$ is nearly identical, so I will omit it.

- **Triangle inequality:** First suppose that $|x+a| \geq |y+b|$. Then

$$\begin{aligned}\|(x+a, y+b)\| &= \max\{|x+a|, |y+b|\} \\ &= |x+a| \\ &\leq |x| + |a| \qquad \text{(by the triangle inequality for the absolute value)} \\ &\leq \max\{|x|, |y|\} + \max\{|a|, |b|\} \\ &= \|(x,y)\| + \|(a,b)\|.\end{aligned}$$

The proof in the case $|x+a| \leq |y+b|$ is nearly identical, so I will omit it.

**3.** Let $V$ be an inner product space with inner product $(\cdot, \cdot)$. Suppose $u, v \in V$ such that $\|u\| = \|v\| = 1$ and $(u,v) = 1$. Prove that $u = v$.

**Solution:** The way to do this is to show that $(u-v, u-v) = 0$. Notice that

$$\begin{aligned}(u,u) &= \|u\|^2 = (1)^2 = 1 \\ (v,v) &= \|v\|^2 = (1)^2 = 1.\end{aligned}$$

Then

$$\begin{aligned}(u-v, u-v) &= (u, u-v) - (v, u-v) \\ &= (u,u) - (u,v) - \Big((v,u) - (v,v)\Big) \\ &= (u,u) - (u,v) - \Big(\overline{(u,v)} - (v,v)\Big) \\ &= 1 - 1 - (\overline{1} - 1) \\ &= 1 - 1 - (1 - 1) \\ &= 0.\end{aligned}$$

By definiteness of the inner product, we get $u - v = 0$ and hence $u = v$.

**4.** Let $M_{2\times 2}(\mathbb{R})$ be equipped with the inner product

$$(A, B) = \text{trace}(A^T B).$$

Use the Gram-Schmidt process to find an orthonormal basis for the subspace

$$V = \text{span}\left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 7 \\ 1 & 0 \end{bmatrix}\right) = \text{span}(A_1, A_2, A_3).$$

Recall that the projection of $A$ onto $B$ is defined by

$$\text{proj}_B(A) = \frac{(A, B)}{(B, B)} B.$$

**Solution:** We first want to find an orthogonal (but not necessarily orthonormal) basis $\{B_1, B_2, B_3\}$ using Gram-Schmidt orthogonalization:

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 - \text{proj}_{B_1}(A_2) \\ B_3 &= A_3 - \text{proj}_{B_1}(A_3) - \text{proj}_{B_2}(A_3). \end{aligned}$$

To calculate $\text{proj}_{B_1}(A_2)$ we need to calculate two inner products:

$$(A_2, B_1) = \text{trace}(A_2^T B_1) = \text{trace}\left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix}\right) = \text{trace}\left(\begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}\right) = 2,$$

$$(B_1, B_1) = \text{trace}(B_1^T B_1) = \text{trace}\left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{trace}\left(\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}\right) = 4.$$

We can now calculate $B_2$:

$$\begin{aligned} B_2 &= A_2 - \text{proj}_{B_1}(A_2) \\ &= A_2 - \frac{(A_2, B_1)}{(B_1, B_1)} B_1 \\ &= \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}. \end{aligned}$$

We will need to calculate three more inner products to find $B_3$:

$$(A_3, B_1) = \text{trace}(A_3^T B_1) = \text{trace}\left(\begin{bmatrix} 1 & 1 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{trace}\left(\begin{bmatrix} 2 & 0 \\ 14 & 0 \end{bmatrix}\right) = 2,$$

$$(A_3, B_2) = \text{trace}(A_3^T B_2) = \text{trace}\left(\begin{bmatrix} 1 & 1 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}\right) = \text{trace}\left(\begin{bmatrix} 4 & 3 \\ 0 & 21 \end{bmatrix}\right) = 4 + 21 = 25,$$

$$(B_2, B_2) = \text{trace}(B_2^T B_2) = \text{trace}\left(\begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}\right) = \text{trace}\left(\begin{bmatrix} 16 & 0 \\ 0 & 9 \end{bmatrix}\right) = 16 + 9 = 25.$$

We can now calculate $B_3$:

$$\begin{aligned} B_3 &= A_3 - \text{proj}_{B_1}(A_3) - \text{proj}_{B_2}(A_3) \\ &= A_3 - \frac{(A_3, B_1)}{(B_1, B_1)} B_1 - \frac{(A_3, B_2)}{(B_2, B_2)} B_2 \\ &= \begin{bmatrix} 1 & 7 \\ 1 & 0 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} - \frac{25}{25} \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 7 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 4 \\ -3 & 0 \end{bmatrix}. \end{aligned}$$

To find an orthonormal basis $\{E_1, E_2, E_3\}$ we just need to normalize each $B_k$:

$$\begin{aligned} E_1 &= \frac{B_1}{\|B_1\|} \\ E_2 &= \frac{B_2}{\|B_2\|} \\ E_3 &= \frac{B_3}{\|B_3\|}. \end{aligned}$$

We can immediately calculate $\|B_1\|$ and $\|B_2\|$ using our previous calculations:

$$\begin{aligned} \|B_1\| &= \sqrt{(B_1, B_1)} = \sqrt{4} = 2, \\ \|B_2\| &= \sqrt{(B_2, B_2)} = \sqrt{25} = 5. \end{aligned}$$

To calculate $\|B_3\|$ we first need to calculate $(B_3, B_3)$:

$$(B_3, B_3) = \text{trace}(B_3^T B_3) = \text{trace}\left( \begin{bmatrix} 0 & -3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -3 & 0 \end{bmatrix} \right) = \text{trace}\left( \begin{bmatrix} 9 & 0 \\ 0 & 16 \end{bmatrix} \right) = 9 + 16 = 25,$$

so

$$\|B_3\| = \sqrt{(B_3, B_3)} = \sqrt{25} = 5.$$

We can now find the elements of the orthonormal basis $\{E_1, E_2, E_3\}$:

$$\begin{aligned} E_1 &= \frac{B_1}{\|B_1\|} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ E_2 &= \frac{B_2}{\|B_2\|} = \frac{1}{5} \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{3}{5} \\ \frac{4}{5} & 0 \end{bmatrix} \\ E_3 &= \frac{B_3}{\|B_3\|} = \frac{1}{5} \begin{bmatrix} 0 & 4 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{4}{5} \\ -\frac{3}{5} & 0 \end{bmatrix}. \end{aligned}$$