MTH 342 Worksheet 9

Name: Answer Key **Recitation time:** ________

Show your work for each problem.

1. Consider $\mathbb{R}^3$ as an inner product space with the standard inner product denoted by $(\cdot,\cdot)$. Let E be the subspace of $\mathbb{R}^3$ given by

$$E = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : a - b + c = 0 \right\}.$$

(a) Find an orthonormal basis for E.

Solution: First we need to find a basis for E (not necessarily orthogonal). Notice that we can rewrite E as

$$\begin{aligned} E = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : a - b + c = 0 \right\} &= \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : a = b - c \right\} \\ &= \left\{ \begin{bmatrix} b-c \\ b \\ c \end{bmatrix} : b, c \in \mathbb{R} \right\} \\ &= \left\{ b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : b, c \in \mathbb{R} \right\} \\ &= \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

A basis for E is given by

$$\{u_1, u_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

To find an orthogonal (not necessarily orthonormal) basis $\{w_1, w_2\}$, we apply the Gram-Schmidt process:

$$\begin{aligned} w_1 &= u_1, \\ w_2 &= u_2 - \operatorname{proj}_{w_1}(u_2). \end{aligned}$$

In this case,

$$\operatorname{proj}_{w_1}(u_2) = \frac{(u_2, w_1)}{(w_1, w_1)} w_1 = \frac{(-1)(1) + (0)(1) + (1)(0)}{(-1)(-1) + (1)(1) + (0)(0)} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore

$$w_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(-\frac{1}{2}\right) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

Now $\{w_1, w_2\}$ is an orthogonal (but not orthonormal) basis for E. We can scale any of the vectors without affecting the orthogonality, so we will multiply w_2 by 2 to get the orthogonal basis

$$\{z_1, z_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

To find an orthonormal basis $\{v_1, v_2\}$, we simply normalize w_1 and w_2:

$$v_1 = \frac{z_1}{\|z_1\|} = \frac{z_1}{\sqrt{(z_1, z_1)}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}.$$

$$v_2 = \frac{z_2}{\|z_2\|} = \frac{z_2}{\sqrt{(z_2, z_2)}} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\end{bmatrix} = \frac{\sqrt{6}}{6}\begin{bmatrix}-1\\1\\2\end{bmatrix}.$$

(b) Find the orthogonal projection of $v = \begin{bmatrix}1\\2\\3\end{bmatrix}$ onto E.

Solution: Consider the orthogonal basis

$$\{z_1, z_2\} = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix}\right\}.$$

for E that we found in part (a). Then

$$\begin{aligned}
\text{proj}_E(v) &= \text{proj}_{z_1}(v) + \text{proj}_{z_2}(v) \\
&= \frac{(v, z_1)}{(z_1, z_1)} z_1 + \frac{(v, z_2)}{(z_2, z_2)} z_2 \\
&= \frac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{7}{6}\begin{bmatrix}-1\\1\\2\end{bmatrix} \\
&= \frac{1}{6}\begin{bmatrix}2\\16\\14\end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix}1\\8\\7\end{bmatrix}.
\end{aligned}$$

2. Consider the following set of points in the x-y plane:

$$\{(1,2),(3,4),(4,-2),(5,5),(-1,5)\}.$$

Set up an equation to find the best fit quadratic polynomial for these points. That is, find a matrix $\mathbf{A}$ and a vector $\mathbf{b}$ such that the least squares solution to

$$\mathbf{A}\begin{bmatrix}a\\b\\c\end{bmatrix}=\mathbf{b}$$

defines the best fit polynomial $f(x)=ax^2+bx+c$. Recall that the least squares solution to $\mathbf{A}\mathbf{x}=\mathbf{b}$ is the ordinary solution to $\mathbf{A}^*\mathbf{A}\mathbf{x}=\mathbf{A}^*\mathbf{b}$.

Solution: Given the unknown quadratic polynomial ax^2+bx+c, we want to choose $a,b,c\in\mathbb{R}$ to minimize the quantity

$$\sum_{k=1}^{n}|ax_k^2+bx_k+c-y_k|^2.$$

This is equivalent to minimizing

$$\left\|\begin{bmatrix}x_1^2 & x_1 & 1\\ x_2^2 & x_2 & 1\\ \vdots & \vdots & \vdots\\ x_n^2 & x_n & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}-\begin{bmatrix}y_1\\y_2\\ \vdots\\ y_n\end{bmatrix}\right\|.$$

That is, we want to find the least squares solution to

$$\mathbf{A}\begin{bmatrix}a\\b\\c\end{bmatrix}=\mathbf{b}$$

where

$$\mathbf{A}=\begin{bmatrix}1^2 & 1 & 1\\ 3^2 & 3 & 1\\ 4^2 & 4 & 1\\ 5^2 & 5 & 1\\ (-1)^2 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\ 9 & 3 & 1\\ 16 & 4 & 1\\ 25 & 5 & 1\\ 1 & -1 & 1\end{bmatrix}$$

and

$$\mathbf{b}=\begin{bmatrix}1\\4\\-2\\5\\5\end{bmatrix}.$$

To find a,b,c, you need to solve

$$\mathbf{A}^*\mathbf{A}\begin{bmatrix}a\\b\\c\end{bmatrix}=\mathbf{A}^*\mathbf{b}.$$

This is a standard augmented matrix row-reduction exercise, so I will leave it to you to finish.

3. Let $B = A^*A$ where

$$A = \begin{bmatrix} 1 & i \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of B. Is B orthogonally diagonalizable (under the standard inner product $(x, y) = x_1\overline{y_1} + x_2\overline{y_2}$ on $\mathbb{C}^2$)?

Solution: First calculate A^*:

$$A^* = \begin{bmatrix} 1 & 1 & 0 \\ -i & 0 & 1 \end{bmatrix}.$$

Now

$$B = A^*A = \begin{bmatrix} 1 & 1 & 0 \\ -i & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}.$$

Calculating the eigenvalues and eigenvectors of B should be a standard exercise by now, so I will omit the details. You should get eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$ with associated eigenspaces

$$E_1 = \left\{ a \begin{bmatrix} -i \\ 1 \end{bmatrix} : a \in \mathbb{C} \right\}$$

and

$$E_3 = \left\{ b \begin{bmatrix} i \\ 1 \end{bmatrix} : b \in \mathbb{C} \right\}.$$

Notice that the geometric multiplicity matches the algebraic multiplicity for each eigenvalue, so B is diagonalizable. To see that it is *orthogonally* diagonalizable, note that

$$\left(\begin{bmatrix} -i \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix} \right) = (-i)(\bar{i}) + (1)(\bar{1}) = (-i)(-i) + (1)(1) = -1 + 1 = 0,$$

so these eigenvalues are orthogonal.

We didn't need to calculate the eigenspaces to know that B was orthogonally diagonalizable. Notice that B satisfies

$$B^* = (A^*A) = A^*(A^*)^* = A^*A = B,$$

so B is self-adjoint. It is a fact that every self-adjoint matrix is orthogonally diagonalizable.

4. Let V be the set of continuous functions from $[0,1]$ to $\mathbb{R}$. Equip V with the inner product

$$(f(x), g(x)) = \int_0^1 f(x)g(x)\,dx.$$

Let $E = \text{span}\{1, x\}$. Find $\text{proj}_E(g)$ where $g(x) = \dfrac{1}{x^2+1}$.

Solution: First we need to find an orthogonal basis for E by applying the Gram-Schmidt process. We start with the basis

$$\{w_1, w_2\} = \{1, x\}.$$

Our orthogonal basis is $\{v_1, v_2\}$ where

$$v_1 = w_1,$$
$$v_2 = w_2 - \text{proj}_{v_1}(w_2).$$

We need to calculate two inner products:

$$(w_2, v_1) = (x, 1) = \int_0^1 x\,dx = \frac{1}{2}$$

and

$$(v_1, v_1) = \int_0^1 1\,dx = 1.$$

Therefore

$$v_2 = w_2 - \text{proj}_{v_1}(w_2) = x - \frac{(x,1)}{(1,1)}1 = x - \tfrac{1}{2},$$

so our orthogonal basis for E is

$$\{v_1, v_2\} = \{1, x - \tfrac{1}{2}\}.$$

To calculate $\text{proj}_E(g)$ we will need to calculate a few more inner products:

$$(g, v_1) = \int_0^1 \frac{1}{x^2+1}\,dx = \tan^{-1}(x)\Big|_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\begin{aligned}(g, v_2) &= \int_0^1 \frac{x}{x^2+1} - \frac{1}{2}\cdot\frac{1}{x^2+1}\,dx \\ &= \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\tan^{-1}(x)\Big|_0^1 \\ &= \frac{1}{2}\ln(2) - \frac{\pi}{8}\end{aligned}$$

$$(v_2, v_2) = \int_0^1 (x - \tfrac{1}{2})^2\,dx = \frac{1}{12}.$$

We can now calculate $\text{proj}_E(g)$:

$$\begin{aligned}\text{proj}_E(g) &= \frac{(g, v_1)}{(v_1, v_1)}v_1 + \frac{(g, v_2)}{(v_2, v_2)}v_2 \\ &= \frac{\pi}{4}v_1 + \frac{\frac{1}{2}\ln(2) - \frac{\pi}{8}}{\frac{1}{12}}v_2 \\ &= \frac{\pi}{4} + \left(6\ln(2) - \frac{3}{2}\pi\right)(x - \tfrac{1}{2}).\end{aligned}$$