# Problem 1.

Let $f(t) = \frac{1}{1-t}$.

a. Derive a formula for the $n$'th Taylor polynomial around $t_0 = 0$ called $p_n$, of $f$.

b. How large should $n$ be so that $f$ can be approximated by its $n$'th Taylor polynomial with error not exceeding $\epsilon = 10^{-4}$ for all $-1/3 < t < 1/3$?

## Solution

Taylor's theorem tells us that if $f$ is $n$ times differentiable, then $f$ can be expressed as $f(t) = \sum_{k=0}^{n} \frac{(t-a)^k (f(t_0))}{k!} + R_n(t)$ where $R_n(t)$ is the remainder term. As the domain of interest excludes 1, $f$ has an infinity of derivatives. Define

$$p_n(t) = \sum_{k=0}^{n} (t-t_0)^k \frac{f^{(k)}(t_0)}{k!}$$

We need to determine the sequence of coefficients $f^{(n)}(t_0)$ in terms of $n$.

$$\begin{aligned}
f(t_0) =& \frac{1}{1-t_0} && \implies && f(0) = 1\\
f'(t_0) =& \frac{1}{(1-t_0)^2} && \implies && f'(0) = 1\\
f''(t_0) =& \frac{2}{(1-t_0)^3} && \implies && f''(0) = 2\\
f^{(3)}(t_0) =& \frac{2\cdot 3}{(1-t_0)^4} && \implies && f^{(4)}(0) = 2\cdot 3\\
f^{(4)}(t_0) =& \frac{2\cdot 3\cdot 4}{(1-t_0)^5} && \implies && f^{(4)}(0) = 2\cdot 3\cdot 4\\
f^{(5)}(t_0) =& \frac{2\cdot 3\cdot 4\cdot 5}{(1-t_0)^6} && \implies && f^{(5)}(0) = 2\cdot 3\cdot 4\cdot 5\\
f^{(6)}(t_0) =& \frac{2\cdot 3\cdot 4\cdot 5\cdot 6}{(1-t_0)^7} && \implies && f^{(6)}(0) = 2\cdot 3\cdot 4\cdot 5\cdot 6\\
&\vdots && && \vdots\\
f^{(n)}(t_0) =& \frac{n!}{(1-t_0)^{(n+1)}} && \implies && f^{(n)}(0) = n!
\end{aligned}$$

We can then collect this into $p_n$ as

$$p_n(t) = \sum_{k=0}^{n} (t-0)^k \frac{k!}{k!} \implies p_n(t) = \sum_{k=0}^{n} t^k$$

To compute an error bound, observe that this is in fact a geometric sum of $t$, so

$$f(t) = \sum_{k=0}^{\infty} t^k = \sum_{k=0}^{n} t^k + \frac{t^{n+1}}{1-t} = p_n(t) + R_n(t)$$

which gives an explicit error as $R_n(t) = \frac{t^{n+1}}{1-t}$. We could alternatively compute an error bound using the Lagrange's error term.

$$|R_n(x)| = \left| \frac{(n+1)!}{(1-c)^{n+2}} \frac{1}{(n+1)!} x^{n+1} \right| = \left| \frac{x^{n+1}}{(1-c)^{n+2}} \right| \le \frac{|x|^{n+1}}{(2/3)^{n+2}} = \frac{3}{2} \left( \frac{3}{2}|x| \right)^{n+1} \le \frac{3}{2} \frac{1}{2^{n+1}}$$

Here we have used the fact that $|x| \le 1/3$. To make sure that $|R_n(x)| \le 10^{-4}$, we only need

$$\frac{3}{2}\frac{1}{2^{n+1}} < 10^{-4}$$

By calculator, we see that $n \ge 13$ will do it. Thus, $n = 13$ is a value of $n$ such that we can guarantee the error is under the given threshold.

Note that for the problem as initially posed on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ requires us to use the explicit error found above (using the geometric series).

# Problem 2.

Let $g(x) = \frac{1}{2+3x}$.

a. Derive a formula for the $n$'th Taylor polynomial around $x_0 = 0$ called $q_n$, of $g$.

b. How large should $n$ be so that $g$ can be approximated by its $n$'th Taylor polynomial with error not exceeding $\epsilon = 10^{-5}$ for all $0 < x < 1/5$?

## Solution

We will use a substitution shortcut to express $g$ in relation to $f$, rather than re-deriving the series. Note that

$$g(x) = \frac{1}{2+3x} = \frac{1}{2}\frac{1}{1-\left(-\frac{3x}{2}\right)}$$

So with $y = -3x/2$,

$$g(x) = \frac{1}{2}f(y) = \frac{1}{2}\left(p_n(y) + R_n(y)\right) = \frac{1}{2}\sum_{k=0}^{n}\left(-\frac{3x}{2}\right)^k + \frac{1}{2}R_n\left(\frac{-3x}{2}\right)$$

So we get $g(x) = q_n(x) + E_n(x)$ where

$$q_n(x) = \frac{1}{2}\sum_{k=0}^{n}\left(-\frac{3x}{2}\right)^k, \quad E_n(x) = \frac{1}{2}R_n\left(-\frac{3x}{2}\right).$$

$q_n$ is the $n$'th Taylor polynomial and $E_n$ is the remainder of function $g$. By Part (a), we know that

$$R_n(y) = \frac{y^{n+1}}{(1-c)^{n+2}}$$

for some $c$ in between 0 and $y$. Now with $y = -3x/2$,

$$R_n\left(-\frac{3x}{2}\right) = \frac{\left(-\frac{3x}{2}\right)^{n+1}}{(1-c)^{n+2}}$$

for some $c$ in between 0 and $-3x/2$. Because $x$ varies between 0 and $1/5$, $c$ is somewhere between $-3/10$ and 0. We estimate the size of the error term as follows.

$$|E_n(x)| = \frac{1}{2}|R_n\left(-\frac{3x}{2}\right)| = \frac{1}{2}\left(\frac{3x}{2}\right)^{n+1}\frac{1}{(1-c)^{n+2}} \le \frac{1}{2}\left(\frac{3/5}{2}\right)^{n+1}.$$

Here we have used the fact that $1-c$ is at least 1, and $x$ is at most $1/5$. To guarantee that $|E_n(x)| < 10^{-5}$, we only need to choose $n$ such that

$$\frac{1}{2}\left(\frac{3/5}{2}\right)^{n+1} < 10^{-5}.$$

By simply trying with a calculator, we see that $n \ge 8$ is sufficient.

# Problem 3.

Let $h(x) = \frac{1}{1+x^2}$.

a. Derive a formula for the $n$'th Taylor polynomial around $x_0 = 0$ called $r_n$, of $h$.

b. How large should $n$ be so that $h$ can be approximated by its $n$'th Taylor polynomial with error not exceeding $\epsilon = 10^{-5}$ for all $-0.4 < x < 0.5$?

## Solution

We use a similar substitution method rather than finding the $n$-th derivative of $h$.

$$h(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = f\left(-x^2\right)$$

We have

$$h(x) = f(-x^2) = p_n(-x^2) + R_n(-x^2) = \underbrace{\sum_{k=0}^{n} \left(-x^2\right)^k}_{r_{2n}(x)} + \underbrace{R_n(-x^2)}_{Q_{2n}(x)}.$$

We have found the $(2n)$'th Taylor polynomial of $h$:

$$r_{2n}(x) = p_n\left(-x^2\right) = \sum_{k=0}^{n} \left(-x^2\right)^k = \sum_{k=0}^{n} (-1)^k x^{2k} \tag{1}$$

The remainder is $Q_{2n}(x) = R_n(-x^2)$. We also observe that $r_{2n+1}(x) = r_{2n}(x)$ because all odd powers of $x$ has coefficient equal to 0. By Part (a), we know that

$$R_n(y) = \frac{y^{n+1}}{(1-c)^{n+2}}$$

for some $c$ in between 0 and $y$. Now with $y = -x^2$,

$$R_n(-x^2) = \frac{(-x^2)^{n+1}}{(1-c)^{n+2}}$$

for some $c$ in between 0 and $-x^2$. Because $x$ varies between -0.4 and 0.5, $c$ is somewhere between $-0.25$ and 0. We estimate the size of the error term as follows.

$$|Q_{2n}(x)| = \frac{1}{2}|R_n(-x^2)| = \frac{1}{2}(x^2)^{n+1}\frac{1}{(1-c)^{n+2}} \leq \frac{1}{2}(0.25)^{n+1}.$$

Here we have used the fact that $1-c$ is at least 1, and $x^2$ is at most 0.25. To guarantee that $|Q_{2n}(x)| < 10^{-5}$, we only need to choose $n$ such that

$$\frac{1}{2}(0.25)^{n+1} < 10^{-5}.$$

By simply trying with a calculator, we see that $n \geq 7$ is sufficient.

# Problem 4.

Write Matlab code using either a "for" loop or a "while" loop to compute the following sum:

$$\sum_{k=1}^{10} \pi^k \prod_{j=1}^{k} \frac{2j-1}{2j}$$

## Solution

```
format long
total = 0;
for ii = 1:10
        prod = 1;
        for jj = 1:ii
                prod = prod*(2*jj-1)/(2*jj);
        end
        disp(total)
        total = total + pi^ii * prod;
end
disp(total)
```

Which prints

```
0

1.570796326794897

5.271897977203405

14.961359439797098

41.596657769407130

1.169061895720350e+02

3.337820721070890e+02

9.664509171833797e+02

2.829814475789768e+03

8.358525449880932e+03

2.485903534090570e+04
```

So

$$2.485903534090570 \times 10^4 \approx \sum_{k=1}^{10} \pi^k \prod_{j=1}^{k} \frac{2j-1}{2j}$$