

## Homework 4

Due 02/14/2020

1. In this problem, we will approximate the value of  $\sqrt[3]{10}$  in a manner that can be carried out by hand (i.e. using only addition, subtraction, multiplication, division of integers). The allowed error is  $\epsilon = 10^{-3}$ . You can use pocket calculator to do the calculations and verify your results.

- (a) Find a function  $f(x)$  that receives  $\sqrt[3]{10}$  as a root.
- (b) Use bisection method to approximate this root of  $f(x)$ . You will need to answer questions such as: What should the initial interval be? How many steps should be done? What the approximate value of  $\sqrt[3]{10}$  ?
- (c) Use Newton's method to approximate this root of  $f(x)$ . You will need to answer questions such as: what should  $x_0$  be? What is the iteration formula? Although Newton's method does not tell us beforehand the number of iterations needed to get an approximate value within the permitted error, you can use an 'artificial' criteria to stop the iteration: *the iteration process will stop at step  $n$  if  $|x_n - x_{n-1}| < 10^{-4}$ .*

2. Consider a sequence defined recursively as follows.

$$x_{n+1} = \frac{15x_n^2 + 13}{4x_n} - 6, \quad x_0 = 2.$$

- (a) Use your pocket calculator to guess the limit of this sequence. Then use the recursive formula to verify that this number is truly a limit of  $(x_n)$ .
- (b) Find the order of convergence. If the order of convergence is 1, find the linear rate of convergence.
- (c) Do Part (a) and (b) for the sequence

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}, \quad x_0 = 1.$$

3. Consider the function  $f(x) = x^3 - 3x^2 + 1$ . In this problem, we will use Newton's method to approximate all roots of this function.

- (a) Use Matlab to plot this function on the interval  $[-1, 3]$ . How many roots does  $f$  have on this interval? Can  $f$  have any roots outside of this interval?
- (b) Write the iteration formula of the Newton's method.
- (c) Let us label the roots of  $f$  as  $r_1 < r_2 < r_3$ . What is the range of values of  $x_0$  that will guarantee that  $x_n$  converges to  $r_1$  ? The same question for  $r_2$  and  $r_3$ . You can use an applet for Newton's method found on this website <https://www.geogebra.org/m/DGFGBJyU> to help you with this problem.

4. In this problem, we will use Newton's method to find approximate solutions of the system

$$\begin{cases} x^2 + y &= xy, \\ y^2 + x &= y + 1. \end{cases}$$

- (a) Write an iteration formula of the Newton's method.
- (b) Write a Matlab program that allows you adjust the initial point  $(x_0, y_0)$  and the number of steps  $n$ . Then experiment with  $(x_0, y_0) = (1, -1)$  and  $n = 5$ . What do you get?

- (c) Find approximate solution  $(x, y)$  such that  $x, y < 0$ .
- (d) Use experiment to search for a solution  $(x, y)$  such that  $x, y > 0$ . What do you observe?