Problem 1.

Consider the function $f(x) = \frac{1}{4}(5-x)^2$.

A. Solve for all fixed points of f.

B. Write an iteration formula for the fixed point method.

C. With $x_0 = 0.8$, use your pocket calculator to guess the limit of the sequence (x_n). Then use the iteration formula to verify that this number is truly a limit of (x_n).

D. Sketch a cobweb diagram that illustrates the fixed point method with $x_0 = 0.8$.

E. Find the order of convergence. If the convergence is linear (order of convergence is 1), find the linear rate of convergence.

Solution

$$f(x) = x \implies 4x = (5-x^2) \implies (x-1)(x+5) = 0 \implies x = 1,\ -5$$

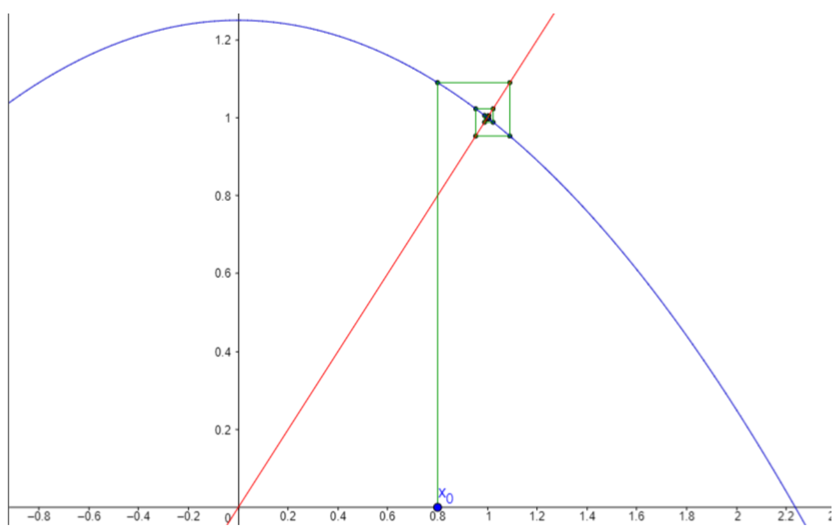
The two fixed points of f are $1, -5$.

The associated iteration formula is $x_{n+1} = f(x_n) = \frac{1}{4}(5-x_n^2)$ for $n = 0, 1, 2, \ldots$. With a pocket calculator, we guess that the limit of the sequence is 1 (after about 10 iterations we see good numerical convergence).

Next, define $a = \lim_{n\to\infty} x_n$. Then

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{4}(5-x_n^2) \implies a = \frac{1}{4}(5-a^2)$$

We have already solved this equation above to find $a = 1,\ -5$. Next, we can sketch the cobweb diagram for this iteration scheme and obtain an image similar to the following:



Next, we find the order of convergence,

$$x_{n+1} - 1 = \frac{1}{4}(5-x_n^2) - 1 = \frac{1}{4}(5-4-x_n^2) = \frac{1-x_n^2}{4}$$

Then

$$|x_{n+1} - 1| = \left|\frac{(x_n-1)(x_n+1)}{4}\right|$$

as $x_n \to 1$, $x_n + 1 \approx 2$, then

$$|x_{n+1} - 1| = \left|\frac{2}{4}(x_n - 1)\right|$$

If e_n is the nth error term,

$$e_{n+1} = \frac{e_n}{2}$$

So the convergence is linear (the exponent is implied to be 1) with linear rate constant 1/2.

Problem 2.

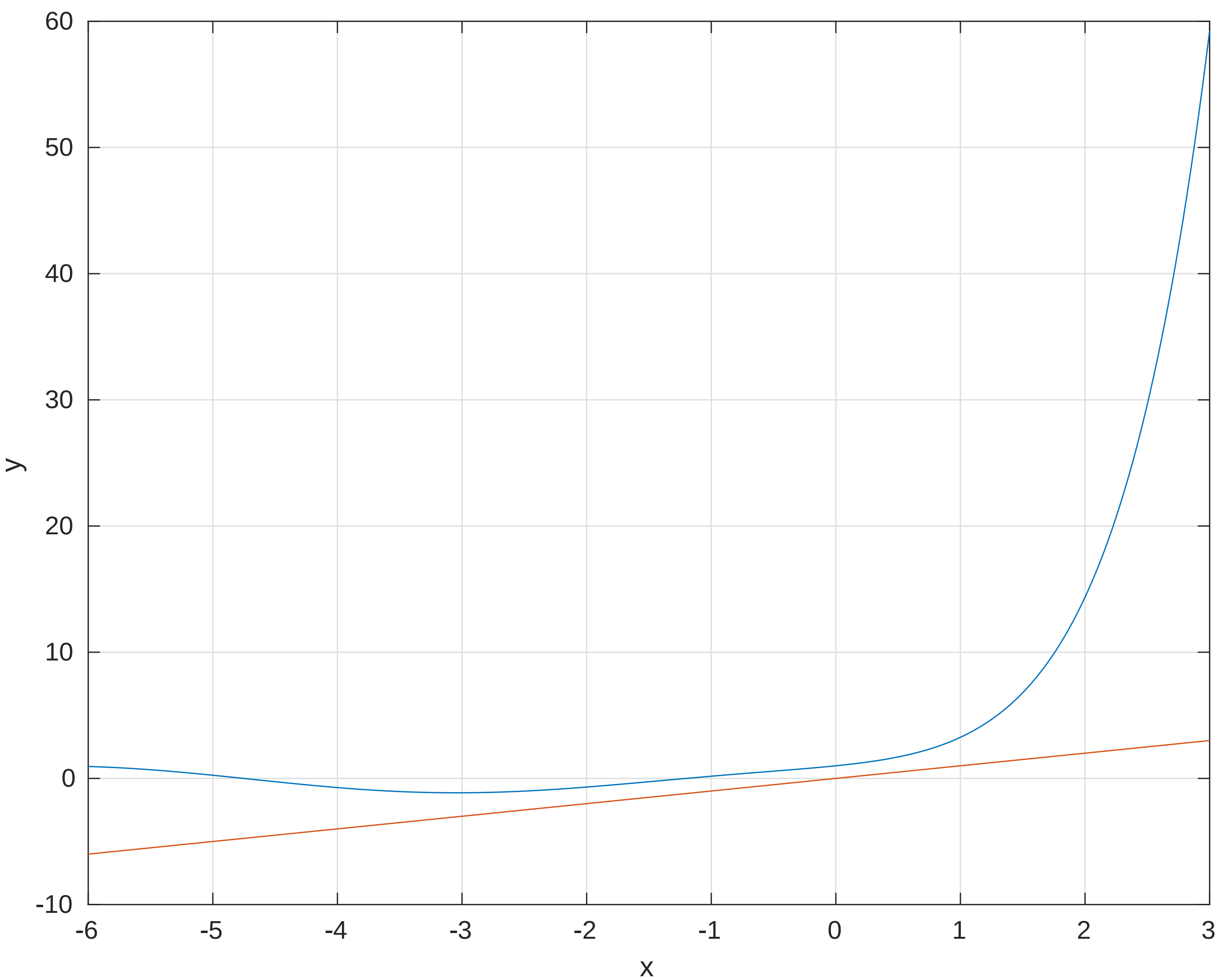
In this problem we will approximate the largest root of the function $f(x) = xe^x + \cos(x)$.

A. Use Matlab to sketch the graph of the function f.

B. Use Newton's method to approximate the largest root of f. You will need to provide the initial estimate x_0, the iteration formula, and the stopping criterion.

C. Use a fixed point method to approximate the largest root of f. You will need to cast the problem as a fixed point problem for some function g and provide the iteration formula, initial guess x_0, and the stopping criterion.

Solution



We see that the blue line (f) has a root between -2 and 0. After 1, the exponential dominates the trigonometric function and there are no additional roots.

```
x_vals = linspace(-6,3, 300);
f = @(x) x.*exp(x) + cos(x);
y_vals = f(x_vals);

f1 = figure();
plot(x_vals, y_vals)
hold on
plot(x_vals, x_vals)
grid on
xlabel('x')
ylabel('y')
hold off
saveas(f1, 'MTH_351_HW5A_fig1', 'epsc')
```

Newton's Method

We start with an initial guess of -1. Newton's method gives the recursive formula

$$x_{n+1} = x_n - \frac{x_n e^{x_n} + \cos(x_n)}{e^{x_n}(x_n+1) - \sin(x_n)}$$

We use an artificial stopping criterion, when $|x_n - x_{n+1}| < 10^{-5}$ we expect to stop at a good estimate, as the value of $f(x_{n+1})$ is very small relative to the gradient. You may simplify this, but there is not significant analytical simplification to be gained here.

We can iterate until we arrive at the stopping criterion and find $x_7 = -1.201060600734212$. We can check a few more iterates and see that x_7 is a good approximation of the true root.

Fixed Point Methods

To convert this to a fixed point problem, we can use a standard trick (add $0 = x - x$) and obtain

$$0 = xe^x + cos(x) + x - x \implies -x = xe^x + cos(x) - x \implies x = -(xe^x + \cos(x) - x)$$

Then

$$x = g(x) = -xe^x - \cos(x) + x$$

Then the fixed points of g are the roots of f. We can construct an iteration scheme as

$$x_{n+1} = g(x_n) = x_n - x_n e^{(x_n)} - \cos(x_n)$$

We can reuse the stopping condition from above, and stop when $|x_{n+1} - x_n| < 10^{-5}$.

With $x_0 = 0$, we can iterate the formula with a calculator and find that after 7 iterations our iteration scheme produces values that satisfy our stopping condition, at $x_7 = -1.201059598235490$, which is very close to the root we found with Newton's method.

Problem 3.

Let α be a root of the function f. In this problem we will investigate the case when $f'(\alpha) = 0$, the convergence of newton's method may drop. For this problem, consider $f(x) = x^2$.

- Sketch a diagram that illustrates the Newton's method.
- Write the iteration formula for Newton's method.
- What is the limit of the sequence(x_n)? What is the order of convergence?

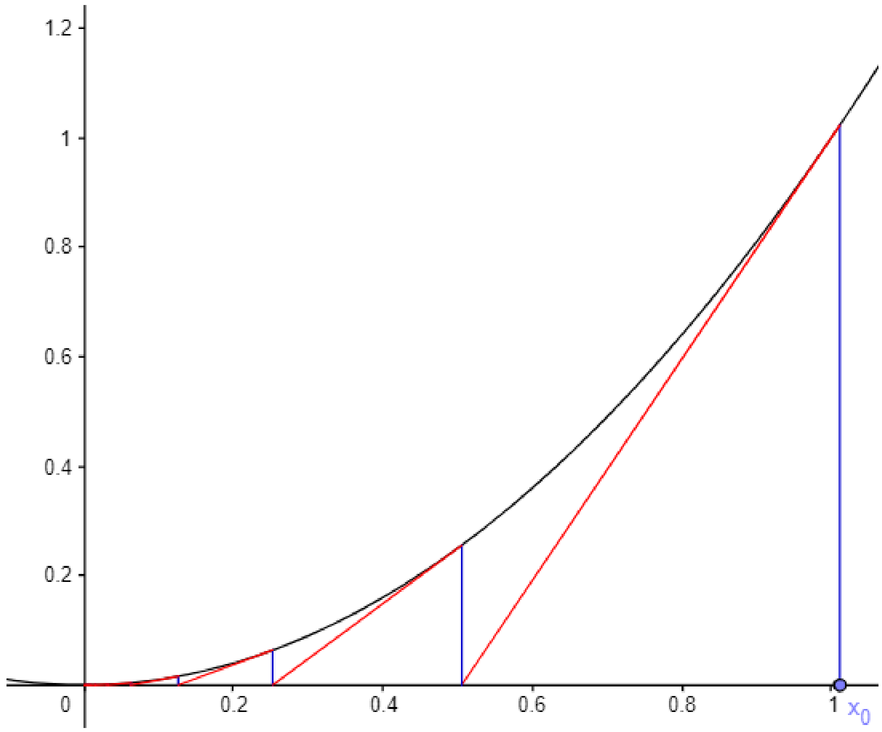


Figure 1: Your Newton method's diagram should resemble this. Please do by hand.

Solution

See figure 1 for a reference cobweb diagram. It appears to converge to $x = 0$ The iteration formula here is given by

$$\epsilon_{n+1} = x_{n+1} = x_n - \frac{x_n^2}{2x_n} = x_n - \frac{x_n}{2} = \frac{x_n}{2}$$

The limit of the sequence (x_n) is 0, which agrees with the cobweb diagram. The rate of convergence can by found in a rather straightforward method as

$$\epsilon_{n+1} = x_{n+1} - 0 = x_{n+1} = \frac{x_n}{2} = \frac{x_n - 0}{2} = \frac{1}{2}\epsilon_n \implies \epsilon_{n+1} = \frac{\epsilon_n}{2}$$

where ϵ_n denotes the error of x_n. Then the convergence is linear with rate constant $\frac{1}{2}$.

Problem 4.

In this problem we want to find a polynomial P whose graph passes through the given points: $(-1, 1),\ (0, -1), (1, 0),\ (2, 2)$.

- Find a polynomial P whose graph passes through the given points. Be sure to simplify P.
- Use Matlab to plot the graph of P on the interval $[-2, 3]$.
- Evaluate P and P' at $x = 1.5$

Solution

We can use Lagrange's method to find an interpolating polynomial. We'll simplify our polynomial a bit before diving into the calculations. Let L_1, L_2, L_3, and L_4 be basis polynomials. Then the interpolating polynomial is

$$P(x) = L_1(x) - L_2(x) + 0L_3(x) + 2L_4(x) = L_1(x) - L_2(x) + 2L_4(x)$$

Then we need to compute L_1, L_2, L_4.

$$L_1(x) = \frac{(x-0)(x-1)(x-2)}{(-1)(-1-1)(-1-2)} = \frac{x(x-1)(x-2)}{-1(-2)(-3)} = \frac{1}{-6}(x)(x-1)(x-2)$$
$$L_2(x) = \frac{(x+1)(x-1)(x-2)}{(1)(-1)(-2)} = \frac{(x+1)(x-1)(x-2)}{2}$$
$$L_4(x) = \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)} = \frac{(x+1)(x-1)(x)}{6}$$

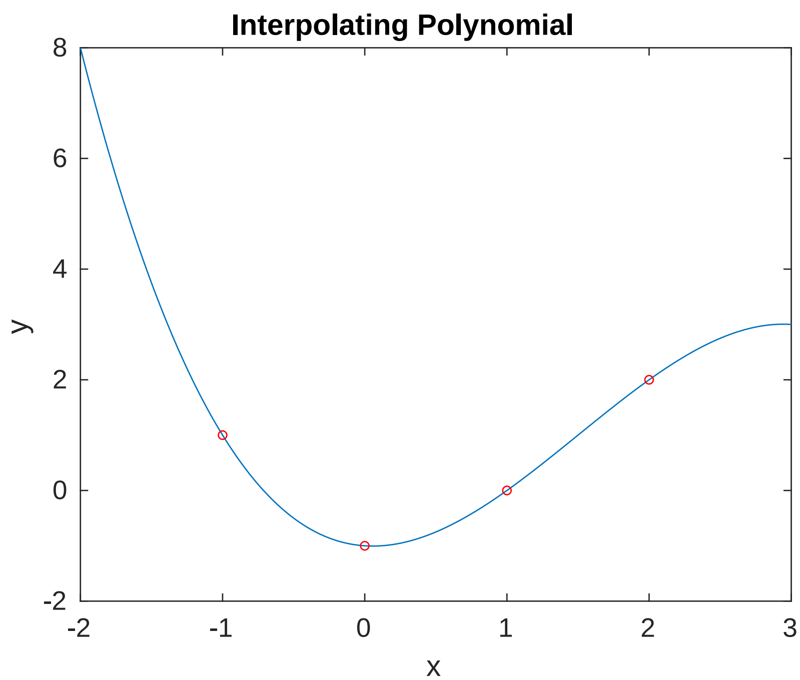
Then

$$P(x) = -\frac{1}{6}(x)(x-1)(x-2) - \frac{1}{2}(x+1)(x-1)(x-2) + \frac{2}{6}(x)(x-1)(x-2)$$

We can simplify this and obtain

$$P(x) = -\frac{1}{3}x^3 + \frac{3}{2}x^2 - \frac{1}{6}x - 1$$

We can plot P and find a plot as



We add in the data points to check that the polynomial interpolates the desired points.

```
x_vals = linspace(-2,3,500);
P = @(x) (- 1/3 .*x.^3 + 3./2.* x.^2 - 1./6.* x - 1);
y_vals = P(x_vals);

%The data points we want to check
x_points = [-1,0,1,2];
y_points = [1,-1,0,2];

%Make the figure
f1 = figure()
plot(x_vals, y_vals)
hold on
```

```
xlabel('x')
ylabel('y')
title('Interpolating Polynomial')
scatter(x_points, y_points, 10,'r')
hold off
saveas(f1, 'MTH_351_HW5A_fig2', 'epsc')
```

We can compute $P(1.5) = 1$. Next

$$P'(x) = -x^2 + 3x - \frac{1}{6} \implies P'(1.5) = \frac{25}{12}$$