Homework 7

Due 03/06/2020

1. Let $f(x) = \frac{1}{x+1}$. For evenly spaced sample points $0 = x_1 < x_2 < \ldots < x_n = 2$, let P_n be the corresponding interpolation polynomial. Find n such that

$$|f(x) - P_n(x)| \leq 10^{-4} \qquad \forall\, x \in [0, 2].$$

2. Given a function f on some interval, say $[-1, 1]$, and an integer $n > 1$, we are interested in the question: *what set of sample points* $\{x_1, x_2, \ldots, x_n\}$ *on* $[-1, 1]$ *should we choose so that the corresponding interpolation polynomial* P_n *can best approximate function* f*?* Note that the number of sample points n is fixed. We are testing different choices of sample points.

   To investigate this question, let us consider an example $f(x) = \frac{1}{1+10x^2}$ and $n = 11$ (i.e. the interval $[-1, 1]$ is partitioned into 10 segments). Consider two different ways of sampling:

   - Evenly spaced $-1 = x_1 < x_2 < \ldots < x_n = 1$,
   - Unevenly spaced $z_k = \cos\left(\frac{2k-1}{2n}\pi\right)$ for $k = 1, 2, \ldots, n$.

   (a) Use the command **Plot** to sketch each set of sample points on the interval $[-1, 1]$.

   (b) Let P_n be the interpolation polynomial corresponding to the set of data points $(x_1, f(x_1))$, $\ldots$, $(x_n, f(x_n))$. Plot P_n and f on the same graph.

   (c) Let Q_n be the interpolation polynomial corresponding to the set of data points $(z_1, f(z_1))$, $\ldots$,$(z_n, f(z_n))$. Plot Q_n and f on the same graph.

   (d) Based on the graphs, is one way of sampling significantly better than the other? Give a rough explanation for your observation.

   (e) The same questions in Parts (b), (c), (d) but for $f(x) = \cos x$.

3. We now test how well quadratic spline interpolation can approximate a function. Consider $f(x) = \frac{1}{1+10x^2}$ on the interval $[-1, 1]$. Choose $n = 11$ evenly spaced sample points $-1 = x_1 < x_2 < \ldots < x_n = 1$. Put $y_j = f(x_j)$. On each subinterval $[x_j, x_{j+1}]$, the function f is approximated by a quadratic polynomial $s_j(x)$ such that the slope of the approximation curve varies smoothly across each sample point. Denote $M_j = s_j'(x_j)$ and $M_{j+1} = s_j'(x_{j+1})$.

   (a) For $j = 1, \ldots, n-1$, use the fact that $s_j(x_j) = y_j$ to write the formula for $s_j(x)$ in terms of y_j, M_j and M_{j+1}.

   (b) For $j = 1, \ldots, n-1$, write an equation that M_j and M_{j+1} has to satisfy such that $s_j(x_{j+1}) = y_{j+1}$.

   (c) In Part (b), we know that each $j = 1, 2, ..., n-1$ yields an equation which M_1, M_2,$\ldots$,M_n has to satisfy. Thus, there are $n-1$ equations to solve for n unknowns M_1, M_2,$\ldots$,M_n. Let us choose $M_1 = 0$ to balance the number of unknowns and the number of equations. Use Matlab is solve for M_2, M_3, $\ldots M_n$.

   (d) Use Matlab to draw the spline interpolation curve (i.e. the concatenation of the quadratic curves s_1, s_2,$\ldots$, s_{n-1}). Plot f and the interpolation P_n (from Part (b) of Problem 1) on the same plot. Which is the better approximation of f (the spline curve or the polynomial curve)?