Denote $I = \int_0^1 \frac{1}{1+4x^2}\,\mathrm{d}x$.

Problem 1.

Find the exact value of I.

Solution

$$\int_0^2 \frac{1}{1+4x^2}\,\mathrm{d}x = \frac{1}{2}\arctan(2x)\Big|_0^1 = \frac{1}{2}(\arctan(2) - \arctan(0)) = \frac{\arctan(2)}{2}$$

With a calculator, $I \approx 0.231823804500403$.

Problem 2.

For a generic positive integer n we take $n+1$ equally spaced sample points indexed by $x_0, x_1, \dots, x_n$ on the interval $[0,1]$. Denote by L_n, R_n, M_n, T_n the Riemann sums corresponding to the left-point, right-point, midpoint, and trapezoid rule. Use sigma notation to write a formula for each L_n, R_n, M_n, T_n.

Solution

Set $x_i = \frac{i}{n}$.

$$L_n = \sum_{i=0}^{n-1} \frac{1}{n} f(x_i) = \sum_{i=0}^{n-1} \frac{1}{n}\frac{1}{1+4x_i^2} = \sum_{i=0}^{n-1} \frac{1}{n}\frac{1}{1+\left(\frac{2i}{n}\right)^2} = \sum_{i=0}^{n-1} \frac{n}{n^2+4i^2}$$

$$R_n = \sum_{i=1}^{n} \frac{1}{n} f(x_i) = \sum_{i=1}^{n} \frac{n}{n^2+4i^2}$$

Note that the indexing has changed between L_n and R_n.

$$\frac{x_i + x_{i+1}}{2} = \frac{1}{2}\frac{i}{n} + \frac{1}{2}\frac{i+1}{n} = \frac{2i+1}{2n} \implies 4\left(\frac{2i+1}{2n}\right)^2 = \left(\frac{2i+1}{n}\right)^2$$

$$M_n = \sum_{i=0}^{n-1} \frac{1}{n} f\left(\frac{x_i + x_{i+1}}{2}\right) = \sum_{i=0}^{n-1} \frac{1}{n}\frac{1}{1+\left(\frac{2i+1}{n}\right)^2} = \sum_{i=0}^{n-1} \frac{1}{n + \frac{1}{n}(2i+1)^2} = \sum_{i=0}^{n-1} \frac{n}{n^2+(2i+1)^2}$$

$$T_n = \sum_{i=0}^{n-1} \frac{1}{n}\frac{f(x_i)+f(x_{i+1})}{2} = \sum_{i=0}^{n-1} \frac{1}{2n}\left(\frac{1}{1+\left(\frac{2i}{n}\right)^2} + \frac{1}{1+\left(\frac{2i+2}{n}\right)^2}\right)$$

Problem 3.

Which of these three methods gives the best approximation of I when $n = 4$?

Solution

We can use the above formulas to compute the approximations, then find the error bounds.

$$\begin{aligned} |L_4 - I| &\approx 0.098348718026032 \\ |R_4 - I| &\approx 0.101651281973968 \\ |M_4 - I| &\approx 0.101651281973968 \\ |T_4 - I| &\approx 0.001651281973968 \end{aligned}$$

T_4 gives the best approximation of the four choices. The left, right, and midpoint approximations are of similar quality.

Problem 4.

Write Matlab code to compute L_n, R_n, M_n, and T_n for $n = 8, 16, 32, 64$.

Solution

We do this in Matlab. For simplicity, we write one script for each method which allows us to easily change n (line 3). Both the approximation and error are printed to the console.
Left point method:

```
1  a = 0;
2  b = 1;
3  n = 4;
4  xvals = linspace(a,b,n+1);  % Generate n+1 points
5  yvals = objective(xvals);
6  total = 0;
7  for ii = 1:n
8      total = total + (yvals(ii))*(b-a)/n;
9  end
10 disp(total)
11 I = atan(2)/2;
12 error = total - I;
13 disp( abs(error) )
14 function out = objective(in)
15    out = 1./(1 + 4.*in.^2);
16 end
```

Right point method:

```
1  a = 0;
2  b = 1;
3  n = 4;
4  xvals = linspace(a,b,n+1);  % Generate n+1 points
5  yvals = objective(xvals);
6  total = 0;
7  for ii = 1:n
8      total = total + (yvals(ii+1))*(b-a)/n;
9  end
10 disp(total)
11 I = atan(2)/2;
12 error = total - I;
13 disp( abs(error) )
14 function out = objective(in)
15    out = 1./(1 + 4.*in.^2);
16 end
```

Trapezoidal method:

```
1  a = 0;
2  b = 1;
3  n = 4;
4  xvals = linspace(a,b,n+1);  % Generate n+1 points
5  yvals = objective(xvals);
6  total = 0;
7  for ii = 1:n
8      total = total + (yvals(ii) + yvals(ii+1))/2*(b-a)/n;
```

```
end
disp(total)
I = atan(2)/2;
error = total - I;
disp( abs(error) )
function out = objective(in)
   out = 1./(1 + 4.*in.^2);
end
```

Midpoint method:

```
a = 0;
b = 1;
n = 4;
xvals = linspace(a,b,n+1);  % Generate n+1 points
xvals = xvals + 1/n;  % Shift xvalues by one half
yvals = objective(xvals);
total = 0;
for ii = 1:n
    total = total + (yvals(ii))*(b-a)/n;
end
disp(total)
I = atan(2)/2;
error = total - I;
disp( abs(error) )
function out = objective(in)
   out = 1./(1 + 4.*in.^2);
end
```

Problem 5.

Find values of n such that the error for each left point, right point, midpoint, and trapezoidal rule approximations are bounded by $\epsilon = 0.0001$.

Solution

We need to find bounds on K and $\widetilde{K}$.

$$|f'(x)| = \frac{8x}{(1+4x^2)^2} \implies |f'(x)| \le \frac{8(1)}{(1+4(0)^2)^2} = 8, \quad x \in [0,1]$$

So $K \le 8$. We can solve the constrained optimization problem on $[0,1]$ by finding roots of f'' on $[0,1]$.

$$|f''(x)| = \left|\frac{128x^2}{(1+4x^2)^3} - \frac{8}{(1+4x^2)^2}\right| = \frac{96x^2-8}{(1+4x^2)^3} \le \frac{96(1)^2-8}{(1+4(0)^2)^3} = 88.$$

Hence, $\tilde{K} \le 88$.

Now we can compute bounds.

$$e_n^{(L)} \le \frac{(b-a)^2}{2n} K \le \frac{(1-0)^2}{2n} 8 = \frac{4}{n}.$$

To make sure that $e_n^{(L)} < 10^{-4}$, we only need to require that $4/n < 10^{-4}$. We see that any $n > 40000$ will do it. We have

$$e_n^{(M)} \le \frac{(b-a)^3}{24n^2}\tilde{K} \le \frac{(1-0)^3}{24n^2} 88 = \frac{11}{3}\frac{1}{n^2}.$$

To make sure that $e_n^{(M)} < 10^{-4}$, we only need to require that $\frac{11}{3}\frac{1}{n^2} < 10^{-4}$. We see that any $n \geq 192$ will do it. This is a much smaller n than in the left or right point approximations.

$$e_n^{(T)} \leq \frac{(b-a)^3}{12n^2}\tilde{K} \leq \frac{(1-0)^3}{12n^2}88 = \frac{22}{3}\frac{1}{n^2}.$$

To make sure that $e_n^{(T)} < 10^{-4}$, we only need to require that $\frac{22}{3}\frac{1}{n^2} < 10^{-4}$. We see that any $n \geq 271$ will do it. This is still a much smaller n than in the left or right point approximations.