Blowup solutions of a Navier-Stokes-like equation – A probabilistic perspective

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February 19, 2020

NSE in 3D

$$\text{(NSE)}: \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ \text{div } u = 0 & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^3. \end{cases}$$

Translation symmetry :

$$u(x,t) \quad \to \quad u(x - x_0, t)$$

$$\dots$$

Scaling symmetry :

$$\begin{aligned} u(x,t) \quad &\to \quad u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t) \\ p(x,t) \quad &\to \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t) \\ u_0(x) \quad &\to \quad u_{\lambda 0}(x) = \lambda u_0(\lambda x) \\ f(x,t) \quad &\to \quad f_\lambda(x,t) = \lambda^3 f(\lambda x, \lambda^2 t) \end{aligned}$$

Main results

$$(\text{cNSE}) : \begin{cases} \partial_t u - \Delta u = \sqrt{-\Delta}(u^2) & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(\cdot, 0) = \frac{2\gamma}{1+|x|^2} & \text{in } \mathbb{R}^3 \end{cases}$$

Dascaliuc, Orum, Pham (2019)

- For any $\gamma \in \mathbb{R}$, (cNSE) has a solution in $L^5(\mathbb{R}^3 \times (0, T))$ for some $0 < T \leq \infty$.
- If $0 \leq \gamma < 1$ then (cNSE) has a unique solution in $L^5(\mathbb{R}^3 \times (0, \infty))$.
- If $\gamma = 1$ then $u(x, t) = u_0(x)$ is the unique solution in $L^5(\mathbb{R}^3 \times (0, T))$ for every $T < \infty$.
- If $\gamma > \frac{9}{2} e^{8/3} \approx 64.76$ then the solution blows up in finite time.

Diffusion equation – Probabilistic representation

In $\mathbb{R}^d \times (0, \infty)$, consider the initial-value problem

$$\begin{cases} \partial_t u - \frac{1}{2}\Delta u &= 0, \\ u(x,0) &= u_0(x). \end{cases}$$

Classical solution:

$$u(x,t) = \int_{\mathbb{R}^d} \underbrace{\frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|y-x|^2}{2t}\right)}_{\Phi(y-x,t)} u_0(y)dy.$$

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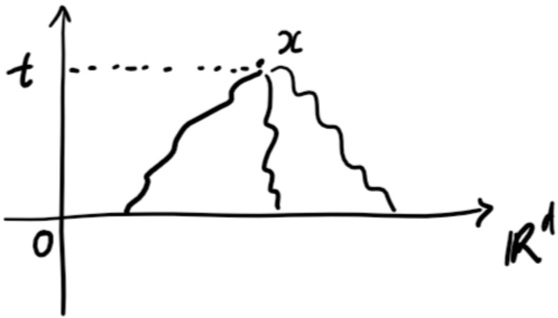
Observe: $\Phi(\cdot - x, t)$ is the p.d.f of an $\mathcal{N}(x, t\, I_d)$-random variable in $\mathbb{R}^d$, e.g. Brownian motion B_t^x.

$$u(x,t) = \mathbb{E}[u_0(B_t^x)].$$

Diffusion equation – Probabilistic representation



Reaction-Diffusion equation – Probabilistic representation

$$\begin{cases} \partial_t u - \frac{1}{2}\Delta u & = & -K(x)u, \\ u(x,0) & = & u_0(x). \\ \end{cases}$$

Feynman-Kac formula (1940s):

$$u(x,t) = \mathbb{E}\left[u_0(B_t^x)\exp\left(-\int_0^t K(B_s^x)ds\right)\right].$$

The problem can be formulated and generalized (with drift term ∇u and forcing f) by Itô calculus (1950s).

KPP-Fisher equation

In $\mathbb{R} \times (0, \infty)$, consider the equation (Kolmogorov-Petrovskii-Piskunov (KPP), Fisher, 1937):

$$\begin{cases} u_t - \frac{1}{2} u_{xx} & = & u^2 - u, \\ u(x,0) & = & u_0(x). \end{cases}$$

With $\Psi = e^{-t}\Phi$,

$$u(x,t) = \int_{\mathbb{R}} \Psi(x-y,t) u_0(y) dy + \int_0^t \int_{\mathbb{R}} \Psi(x-y,s) u^2(y,t-s) dy ds.$$

Noting that Ψ is a p.d.f on $\mathbb{R} \times (0, \infty)$, McKean (1975) gave a probabilistic description of this equation by branching process.

KPP-Fisher equation

$T \sim Exp(1)$: holding time (the clock).

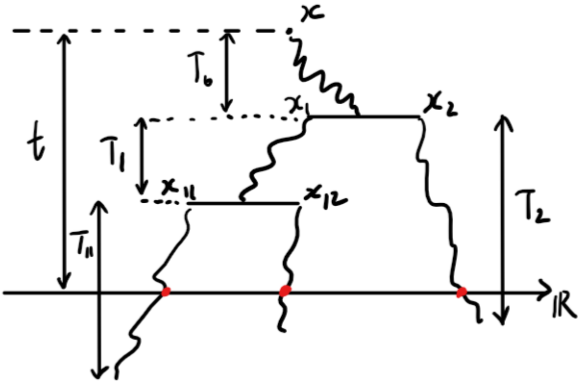
$$u(x,t) = \mathbb{E}\left[u_0(B_t^x)\mathbb{1}_{T>t}\right] + \mathbb{E}\left[u^2(B_T^x, t-T)\mathbb{1}_{T\le t}\right]$$

In other words, $u(x,t) = \mathbb{E}[\mathbf{X}(x,t)]$ where

$$\mathbf{X}(x,t) = \begin{cases} u_0(B_t^x) & \text{if} \quad T > t, \\ \mathbf{X}^{(1)}(B_T^x, t-T)\,\mathbf{X}^{(2)}(B_T^x, t-T) & \text{if} \quad T \le t. \end{cases}$$

Here $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are i.i.d copies of $\mathbf{X}$ and are independent of T.

KPP-Fisher equation – Branching process



$$\mathbf{X}(x,t) = u_0(B^{x_{11}}_{t-T_0-T_1})u_0(B^{x_{12}}_{t-T_0-T_1})u_0(B^{x_2}_{t-T_0}).$$

Diffusion-reaction equation – Fourier domain – Ex. 1

The heat operator $\partial_t - \Delta$ naturally induces a clock in the Fourier domain. For example,

$$u_t - u_{xx} = bu, \quad u(x,0) = u_0(x).$$

In Fourier domain,

$$\hat{u}(\xi,t) = e^{-t\xi^2}\hat{u}_0(\xi) + \int_0^t e^{-s\xi^2} b\hat{u}(\xi, t-s)ds.$$

Put $\chi = \xi^2\hat{u}$. Then

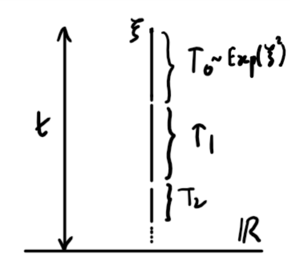
$$\chi(\xi,t) = e^{-t\xi^2}\chi_0(\xi) + \int_0^t \underbrace{\xi^2 e^{-s\xi^2}}_{\text{p.d.f}} \frac{b}{\xi^2}\chi(\xi, t-s)ds.$$

Diffusion-reaction equation – Fourier domain – Ex. 1

$$\chi(\xi, t) = \mathbb{E}[\mathbf{X}(\xi, t)]$$

where

$$\mathbf{X}(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T > t, \\ \frac{b}{\xi^2}\mathbf{X}(\xi, t - T) & \text{if} \quad T \leq t. \end{cases}$$



$$\mathbf{X}(\xi, t) = \left(\frac{b}{\xi^2}\right)^{N_t} \chi_0(\xi), \quad N_t = \inf\{n : T_0 + T_1 + ... + T_n > t\}$$

Diffusion-reaction equation – Fourier domain – Ex. 2

$$u_t - u_{xx} = (\cos x)u, \quad u(x,0) = u_0(x).$$

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)e^{-t\xi^2} + \frac{c}{2}\int_0^t \xi^2 e^{-s\xi^2}\Big(\frac{\hat{u}(\xi-1,t-s)}{\xi^2} + \frac{\hat{u}(\xi+1,t-s)}{\xi^2}\Big)ds$$

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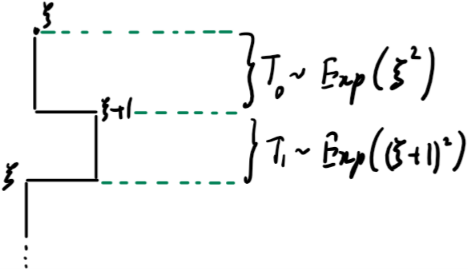

$$\mathbf{X}(\xi,t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T > t, \\ \frac{c}{\xi^2}\mathbf{X}(W, t-T) & \text{if} \quad T \le t. \end{cases}$$

$$\mathbb{P}_\xi(W = \xi - 1) = \mathbb{P}_\xi(W = \xi + 1) = 1/2.$$

Navier-Stokes equations

$$(\text{NSE}) : \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^d \times (0,\infty), \\ \text{div } u = 0 & \text{in} \quad \mathbb{R}^d \times (0,\infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^d. \end{cases}$$

Integro-differential equation:

$$u(x,t) = e^{\Delta t} u_0 - \int_0^t e^{\Delta s} \mathbf{P} \text{div}[u(t-s) \otimes u(t-s)] ds.$$

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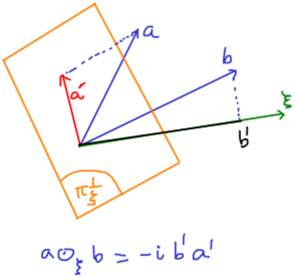
In Fourier domain:

$$
\hat{u}(\xi,t) = e^{-|\xi|^2 t} \hat{u}_0(\xi) + c_0 \int_0^t e^{-|\xi|^2 s} |\xi| \int_{\mathbb{R}^d} \hat{u}(\eta, t-s) \odot_\xi \hat{u}(\xi - \eta, t-s) d\eta ds
$$

where $a \odot_\xi b = -i(e_\xi \cdot b)(\pi_{\xi^\perp} a)$.

Fourier-transformed Navier-Stokes equations (FNS)



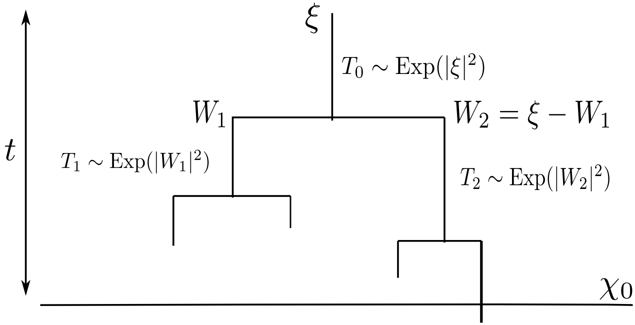
Normalization to (FNS): LJS 1997, Bhattacharya et al 2003.

$$\begin{aligned} \chi(\xi, t) &= e^{-t|\xi|^2}\chi_0(\xi) \\ &+ \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^d} \chi(\eta, t-s) \odot_\xi \chi(\xi-\eta, t-s) H(\eta|\xi) d\eta ds \end{aligned}$$

where $\chi = c_0\hat{u}/h$ and $H(\eta|\xi) = \frac{h(\eta)h(\xi-\eta)}{|\xi|h(\xi)}$.

h: *majorizing kernel*, i.e. $h * h = |\xi| h$.

Cascade structure of FNS

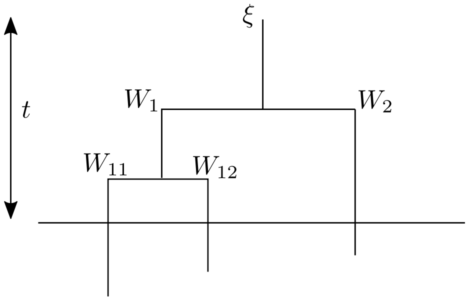


Define a stochastic multiplicative functional recursively as

$$\mathbf{X}_{\text{FNS}}(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T_0 > t, \\ \mathbf{X}^{(1)}_{\text{FNS}}(W_1, t - T_0) \odot_\xi \mathbf{X}^{(2)}_{\text{FNS}}(\xi - W_1, t - T_0) & \text{if} \quad T_0 \le t. \end{cases}$$

An example of $\mathbf{X}_{\mathrm{FNS}}$

Consider the following event:



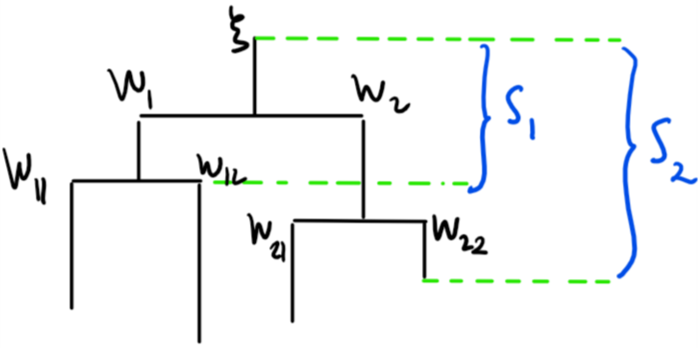
On this event,

$$\mathbf{X}_{\mathsf{FNS}}(\xi, t) = (\chi_0(W_{11}) \odot_{W_1} \chi_0(W_{12})) \odot_\xi \chi_0(W_2).$$

Three ingredients: clocks, branching process, product.
Cascade structure = clocks + branching process.

Stochastic explosion

$$S_n = \min_{|\nu|=n} \sum_{j=0}^{n} T_{\nu|j}, \qquad S = \lim_{n\to\infty} S_n = \sup_{n\in\mathbb{N}} S_n$$

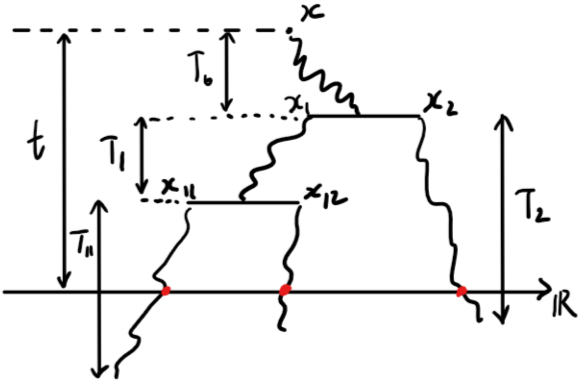


Explosion event: $\{S < \infty\}$.

Non-explosion event : $\{S = \infty\}$.

Examples of non-explosion

$$u_t - u_{xx} = u^2 - u, \quad u(x,0) = u_0(x).$$



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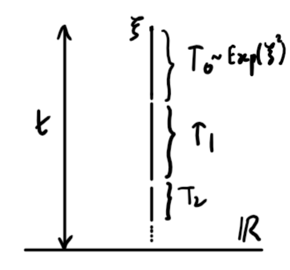

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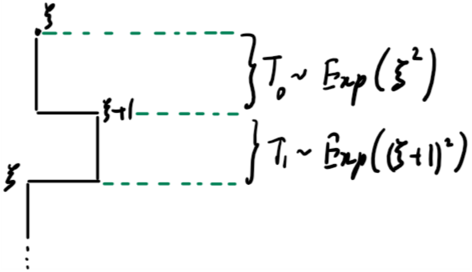
$$u_t - u_{xx} = (\cos x)u, \quad u(x,0) = u_0(x).$$



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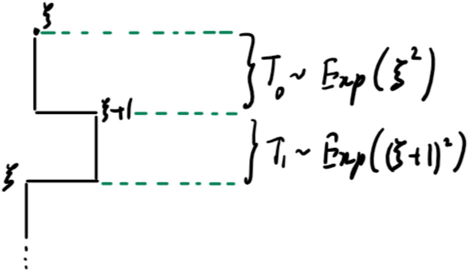

$$\mathbb{P}_\xi(W = \xi - 1) = \mathbb{P}_\xi(W = \xi + 1) = 1/2.$$

$$\sum_{n=1}^{\infty} T_{\nu|n} = \sum_{n=1}^{\infty} \frac{\bar{T}_{\nu|n}}{|W_{\nu|n}|^2} \geq \sum_{k=1}^{\infty} \frac{\bar{T}_{\nu_k}}{\xi^2} = \frac{1}{\xi^2} \sum_{k=1}^{\infty} \bar{T}_{\nu_k} = \infty \qquad a.s.$$

Cheap Navier-Stokes equation

$$\text{(cNSE)}: \begin{cases} \partial_t u - \Delta u = \sqrt{-\Delta}(u^2) & \text{in} \quad \mathbb{R}^d \times (0,\infty), \\ u(\cdot,0) = \gamma \check{h}/c_0 & \text{in} \quad \mathbb{R}^d \end{cases}$$

With $\chi = c_0 \hat{u}/h$, we have

$$\begin{aligned} \chi(\xi,t) &= e^{-t|\xi|^2}\gamma \\ &+ \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^d} \chi(\eta,t-s)\chi(\xi-\eta,t-s)H(\eta|\xi)d\eta ds \\ &= \mathbb{E}[\mathbf{X}(\xi,t)] \end{aligned}$$

where

$$\mathbf{X}(\xi,t) = \begin{cases} \gamma & \text{if} \quad T_0 > t, \\ \mathbf{X}^{(1)}(W_1, t-T_0)\mathbf{X}^{(2)}(\xi - W_1, t-T_0) & \text{if} \quad T_0 \le t. \end{cases}$$

Stochastic explosion

Branching process may never stop, potentially making $\mathbf{X}_{FNS}$ not well-defined.

- Property of cascade structure, not of product.
- Depending on the majorizing kernel h.
- 3D self-similar cascade $h_{\mathrm{dilog}}(\xi) = C|\xi|^{-2}$: stochastic explosion a.s. (Dascaliuc, Pham, Thomann, Waymire 2019)
- 3D Bessel cascade $h_{\mathrm{b}}(\xi) = C|\xi|^{-1}e^{-|\xi|}$: non-explosive a.s. (Orum, Pham 2019)

Cascade solutions

When stochastic explosion happens, how can we define a stochastic cascade solution and is it unique?

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Introducing a *ground state* $\mathbf{X}_0 = \mathbf{X}_0(\xi, t)$:

$$\mathbf{X}_n(\xi, t) = \begin{cases} \gamma & \text{if} \quad T_0 > t, \\ \mathbf{X}^{(1)}_{n-1}(W_1, t - T_0)\mathbf{X}^{(2)}_{n-1}(\xi - W_1, t - T_0) & \text{if} \quad T_0 \leq t. \end{cases}$$

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- If $\gamma = 1$ and $\mathbf{X}_0 = 1$ then $\mathbf{X}_n = 1$ for all n. Thus, $\chi = \lim \mathbb{E}\mathbf{X}_n = 1$.
- If $\gamma = 1$ and $\mathbf{X}_0 = 0$ then $\chi = \lim \mathbb{E}\mathbf{X}_n = \mathbb{P}(S_\xi > t)$.

$$\boxed{\chi(\xi, t) = \sum_{n=1}^{\infty} \gamma^n p_n(\xi, t)}$$

$p_n(\xi, t) = \mathbb{P}(S_\xi > t, \text{ exactly } n \text{ branches cross}).$

Cheap NSE in 3D

Bessel majorizing kernel: $h = h_{\mathrm{b}}(\xi) = \frac{1}{2\pi}\frac{e^{-|\xi|}}{|\xi|}$.

$$(\mathrm{cNSE}): \begin{cases} \partial_t u - \Delta u = \sqrt{-\Delta}(u^2) & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(\cdot, 0) = \frac{2\gamma}{1+|x|^2} & \text{in } \mathbb{R}^3 \end{cases}$$

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If $0 \le \gamma < 1$ then (cNSE) has a unique solution in $L^5(\mathbb{R}^3 \times (0,\infty))$. If $\gamma > \frac{9}{2}e^{8/3} \approx 64.76$ then the solution blows up in finite time.

Cheap NSE in 3D when $\gamma < 1$

$p_n(\xi, t) = \mathbb{P}(S_\xi > t$, exactly n branches cross).
By conditioning on the first time of branching, we get

$$p_n(\xi, t) = \int_0^t |\xi|^2 e^{-s|\xi|^2} \int_{\mathbb{R}^3} \sum_{k=1}^{n-1} p_k(\eta, t-s) p_{n-k}(\xi - \eta, t-s) H(\eta|\xi) d\eta ds$$

By induction, one can prove

$$p_n(\xi, t) \leq \theta \lambda^{n-1} C_n e^{-|\xi|\sqrt{t}},$$

where

- $\theta = e^{1/4}$, $\lambda = 2e^{3/4}$,
- (C_n) is the Catalan sequence

$$\begin{cases} C_1 = 1 \\ C_n = \sum_{k=1}^{n-1} C_k C_{n-k} \end{cases}$$

Cheap NSE in 3D when $\gamma < 1$

For any $0 < \kappa < 1$,

$$p_n(\xi, t) \le \left(\theta \lambda^{n-1} C_n e^{-|\xi|\sqrt{t}}\right)^{\kappa} \lesssim (4\lambda)^{\kappa n} e^{-\kappa|\xi|\sqrt{t}}.$$

If $\gamma < 1$, choose κ small such that $4^{\kappa}\lambda^{\kappa}\gamma < 1$.

$$\chi(\xi, t) = \mathbb{P}(S_\xi > t) = \sum_{n=1}^{\infty} \gamma^n p_n(\xi, t) \lesssim \sum_{n=1}^{\infty} (\underbrace{4^{\kappa}\lambda^{\kappa}\gamma}_{<1})^n e^{-\kappa|\xi|\sqrt{t}}.$$

Cheap NSE in 3D when γ is large

$$q_n(t) = \inf_{1/3 \le |\xi| \le 1} p_n(\xi, t)$$

$$q_n(t) \ge \alpha \int_0^t e^{-(t-s)} \sum_{k=1}^{n} q_k(s) q_{n-k}(s) ds$$

By induction,

$$p_n(\xi, t) \ge q_n(t) \ge \alpha^{n-1} t^{n-1} e^{-nt}.$$

For large γ and for some t,

$$\chi(\xi, t) = \sum_{n=1}^{\infty} \gamma^n p_n(\xi, t) \gtrsim \sum_{n=1}^{\infty} (\underbrace{\gamma \alpha t e^{-t}}_{>1})^n = \infty.$$

Thank You!