Main points in Sections 5.1 and 5.2

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Contents

1 Find the values of a composite function

To find the values of a composite functions $(f \circ g)(x)$ at $x = a$, first we compute $g(a)$ and call the result b; second, we compute $f(b)$. And that is the final result. Why do we do so? That is because $(f \circ g)(x)$ is just a short-hand for $f(g(x))$.
Ex 1 (Problem 11, page 252)
Here we are given $f(x) = 2x$, and $g(x) = 3x^2 + 1$. We are asked to find $(f \circ g)(4)$. The first step is to compute $g(4) = 3 \times 4^2 + 1 = 49$. Then we compute $f(49) = 2 \times 49 = 98$. That is the final result.

When you are familiar with this procedure, you can just write :

$$(f \circ g)(4) = f(g(4)) = f(3 \times 4^2 + 1) = f(49) = 2 \times 49 = 98.$$

Other values of other composite functions are :

$$\begin{aligned}
(g \circ f)(2) &= g(f(2)) = g(2 \times 2) = g(4) = 3 \times 4^2 + 1 = 49, \\
(f \circ f)(1) &= f(f(1)) = f(2 \times 1) = f(2) = 2 \times 2 = 4, \\
(g \circ g)(0) &= g(g(0)) = g(3 \times 0^2 + 1) = g(1) = 3 \times 1^2 + 1 = 4.
\end{aligned}$$

2 Find the domain of a composite map

There are 2 things to remember : first, if $f(x)$ is a fraction, the domain is the whole real line excluded x's which make the denominator zero; second, if $f(x)$ is a square root, the domain is all x's which make the expression inside the square root greater than or equal to 0.

Now suppose that you are to find the domain of the composite function $f \circ g$. There are 4 steps you can follow :

1) Find the domain of $g(x)$, and call it A.

2) Find the domain of $f(x)$, and call it B.

3) Find all x such that $g(x)$ lies outside of B.

4) The domain of $f \circ g$ is A excluded these x's.

Ex 2 (Problem 37, page 253)
Here we are given

$$f(x) = \frac{x}{x-1}, \quad g(x) = -\frac{4}{x}$$

No matter what composite function we are concerning, the first 2 steps must always be done :
Step 1. Find the domain of f.
We see that $f(x)$ is a fraction. And the denominator is zero if and only if $x = 1$. Thus the domain of f is the whole real line except 1.
Step 2. Find the domain of g.
We see that $g(x)$ is a fraction. And the denominator is zero if and only if $x = 0$. Thus the domain of $g(x)$ is the whole real line except 0.

Now to find the domain of $f \circ g$, we continue as follow :
Step 3. Find all x such that $g(x)$ lies outside of the domain of f.
Since the domain of f is everything except 1, $g(x)$ lies outside of it if and only if $g(x) = 1$. Solving the equation $-\frac{4}{x} = 1$ gives us $x = -4$.
Step 4. The domain of $f \circ g$ is just the domain of g excluded $x = -4$. That is the whole realine excluded 0 and -4.

To find the domain of $g \circ f$, we do as follow :
Step 3. Find all x such that $f(x)$ lies outside of the domain of g.
Since the domain of g is everything except 0, $f(x)$ lies outside of it if and only if $f(x) = 0$. Solving the equation $\frac{x}{x-1} = 0$ gives us $x = 0$.
Step 4. The domain of $g \circ f$ is just the domain of f excluded $x = 0$. That is the whole realine excluded 1 and 0.

To find the domain of $f \circ f$, we do as follow :
Step 3. Find all x such that $f(x)$ lies outside of the domain of f.
Since the domain of f is everything except 1, $f(x)$ lies outside of it if and only if

$f(x) = 1$. Solving the equation $\frac{x}{x-1} = 1$ gives us no solution.
Step 4. The domain of $g \circ f$ is just the domain of f. That is the whole realine excluded 1.

To find the domain of $g \circ g$, we do as follow :
Step 3. Find all x such that $g(x)$ lies outside of the domain of g.
Since the domain of g is everything except 0, $g(x)$ lies outside of it if and only if $g(x) = 0$. Solving the equation $-\frac{4}{x} = 0$ gives us no solution.
Step 4. The domain of $g \circ g$ is just the domain of g. That is the whole realine excluded 0.

3 Use graph to determine whether a function is one-to-one

This is a test called *horizontal-line test*. We draw arbitrary horizontal lines on the picture. If one of these line intersects the graph more than one time, then the function is not one-to-one. Otherwise, the function is one-to-one.
Ex 3 (Problems 19,21,24, page 264)
Look at the graph of Problem 19. We see that any horizontal line intersects the graph at at most one point. Thus the function is one-to-one.

Look at the graph of Problem 21. We see that the horizontal line $y = 1$ intersects the graph at two points. Thus the function is not one-to-one.

Look at the graph of Problem 24. We see that the horizontal line $y = 2$ intersects the graph at many points. Thus the function is not one-to-one.

4 Verify that two given functions are inverses of each other

Suppose that we are given two function $f(x)$ and $g(x)$. To show that they are inverses of each other, we will verify two things (and they are equally important). First, compute $f(g(x))$ to see that it equals x. Second, compute $g(f(x))$ to see that it also equals x. During the computation, we may need to exclude some values of x to make everything well-defined.
Ex 4 (Problems 39, page 264)
Here we are give

$$f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{x}$$

First, we will check $f(g(x)) = x$:

$$\begin{aligned} f(g(x)) &= f\left(\frac{1}{x}\right) \quad \text{(Here we have to exclude x = 0)} \\ &= \frac{1}{\frac{1}{x}} = x. \end{aligned}$$

Next, we will check $g(f(x)) = x$:

$$\begin{aligned} g(f(x)) &= g\left(\frac{1}{x}\right) \quad \text{(Here we have to exclude x = 0)} \\ &= \frac{1}{\frac{1}{x}} \\ &= x. \end{aligned}$$

<u>Ex 5</u> (Problems 38, page 264)
Here we are given

$$f(x) = (x-2)^2, \quad g(x) = \sqrt{x} + 2$$

First, we will check $f(g(x)) = x$:

$$\begin{aligned} f(g(x)) &= f(\sqrt{x} + 2) \quad (\text{Here we have to exclude } x < 0) \\ &= (\sqrt{x} + 2 - 2)^2 \\ &= x. \end{aligned}$$

Next, we check $g(f(x)) = x$:

$$\begin{aligned} g(f(x)) &= g\left((x-2)^2\right) \\ &= \sqrt{(x-2)^2} + 2 \\ &= |x-2| + 2 \end{aligned}$$

The final result equals x only if $x - 2 \geq 0$, or equivalently $x \geq 2$. Thus all $x < 2$ must be excluded.

5 Find the inverse of a one-to-one function

To find the inverse of a one-to-one function $f(x)$, we simply solve the equation $x = f(y)$ for y. During the computation, we may need to add some conditions to make everything well-defined.
<u>Ex 6</u> (Problems 56, page 265)
Here we are given $f(x) = x^2 + 9$ with $x \geq 0$. The equation $x = f(y)$ (keep in mind that $y \geq 0$) is

$$x = y^2 + 9$$

Subtracting both sides by 9, we get $x - 9 = y^2$. Since $y \geq 0$, we take the square root both sides and get $\sqrt{x-9} = y$. Here we have to assume $x \geq 9$ so that taking square root is legal. Therefore, the inverse function of f is

$$f^{-1}(x) = \sqrt{x-9}, \quad x \geq 9$$