Main points in Sections 7.1 and 7.2

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Contents

1 What to remember ?

In these sections, we learn the inverse functions of the 6 trigonometric functions. They are $\cos^{-1} x$, $\sin^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$. All what we need to remember is the domain and range of $\cos^{-1} x$, $\sin^{-1} x$, and $\tan^{-1} x$ in the following chart. The graphs below can give you more intuition about these three functions and help you remember the chart.

y	Domain	Range
$\cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$\sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$\tan^{-1} x$	All real numbers	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

2 Compute $\cos^{-1} x$, $\sin^{-1} x$ and $\tan^{-1} x$

If you want to find the exact values of these functions, you can follow the following steps :

1) Put $\theta = \cos^{-1} x$ (respectively $\sin^{-1} x$ or $\tan^{-1} x$). And we are finding the angle θ such that $\cos\theta = x$ (respectively $\sin\theta = x$ or $\tan\theta = x$).

2) Write down the range for θ by looking at the chart.

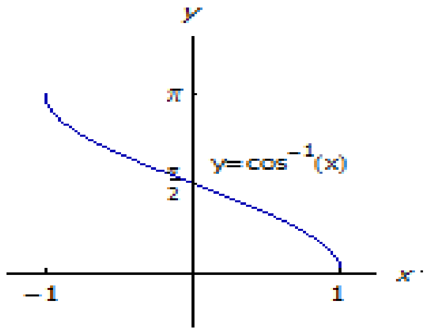

Figure 1: Inverse cosine

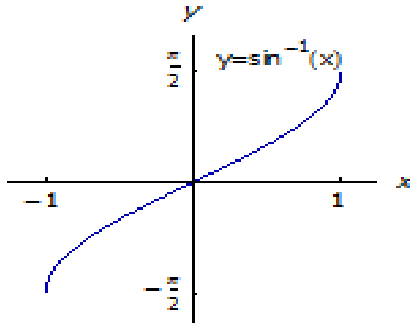

Figure 2: Inverse sine

3) Find θ by using the chart of common values of trigonometric functions.

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	1	0

It is useful to remind yourself that $\sin\theta$ and $\tan\theta$ are odd functions, while $\cos\theta$ is even. One more property of $\cos\theta$ that you will learn later is $\cos(\theta) = -\cos(\pi-\theta)$. That is, two supplementary angles have cosines of opposite signs.
Ex 1 (Problem 19, page 446)
We are to find $\sin^{-1}\frac{\sqrt{2}}{2}$.
Step 1. Put $\theta = \sin^{-1}\frac{\sqrt{2}}{2}$. We are going to find the angle θ such that $\sin\theta = \frac{\sqrt{2}}{2}$.
Step 2. Since we are given the inverse sine, the range of θ is $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
Step 3. By using the chart of common values, we find $\theta = \frac{\pi}{4}$.
Ex 2 (Problem 23, page 446)
We are to find $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

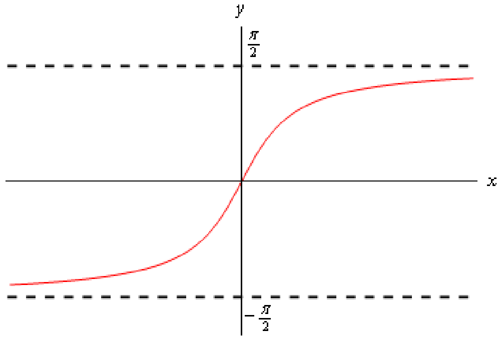

Figure 3: Inverse tangent

Step 1. Put $\theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. We are going to find the angle θ such that $\cos\theta = -\frac{\sqrt{3}}{2}$.
Step 2. Since we are given the inverse cosine, the range of θ is $0 \le \theta \le \pi$.
Step 3. Unfortunately, we do not see in the chart of common values any angle whose cosine is $\frac{\sqrt{3}}{2}$. However, we see that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. Thus, the supplementary angle of $\frac{\pi}{6}$ will have cosine equal $-\frac{\sqrt{3}}{2}$. Therefore,

$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

<u>Ex 3</u> (Problem 37, page 446)
We are to find $\cos^{-1}\left(\cos\frac{4\pi}{5}\right)$.
Step 1. Put $\theta = \cos^{-1}\left(\cos\frac{4\pi}{5}\right)$. We are going to find the angle θ such that $\cos\theta = \cos\frac{4\pi}{5}$.
Step 2. Since we are given the inverse cosine, the range of θ is $0 \le \theta \le \pi$.
Step 3. Since $\frac{4\pi}{5}$ is already in the range, we can pick $\theta = \frac{4\pi}{5}$.
<u>Ex 4</u> (Problem 41, page 446)
We are to find $\sin^{-1}\left(\sin\frac{9\pi}{8}\right)$.
Step 1. Put $\theta = \sin^{-1}\left(\sin\frac{9\pi}{8}\right)$. We are going to find the angle θ such that $\sin\theta = \sin\frac{9\pi}{8}$.
Step 2. Since we are given the inverse sine, the range of θ is $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
Step 3. Here we cannot pick $\theta = \frac{9\pi}{8}$ because $\frac{9\pi}{8}$ exceeds $\frac{\pi}{2}$. Now look at the unit circle, we see that the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ that has the same y is $-\frac{\pi}{8}$. Therefore, $\theta = -\frac{\pi}{8}$.
<u>Ex 5</u> (Problem 45, page 446)
We are to find $\sin\left(\sin^{-1}\frac{1}{4}\right)$.
Put $\theta = \sin^{-1}\frac{1}{4}$. We are going to $\sin\theta$. By the definition of θ, we already have $\sin\theta = \frac{1}{4}$.

3 Compute $\sec^{-1} x$, $\csc^{-1} x$ and $\cot^{-1} x$

If you are asked to find these inverse functions, the first step is the same as mentioned above; the second step is to convert everything to cosine, sine or tangent.

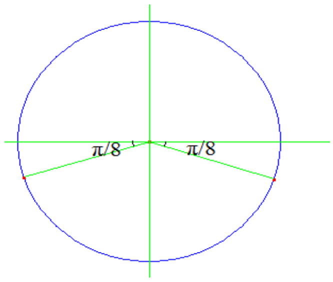

Figure 4: Inverse tangent

Then we return to the problem of finding inverse cosine, sine, and tangent.
<u>Ex 6</u> (Problem 45, page 453)
We are to find $\sec^{-1} 4$.
Put $\theta = \sec^{-1} 4$. We are going to find the angle θ such that $\sec\theta = 4$. That is $\frac{1}{\cos\theta} = 4$. Thus, $\cos\theta = \frac{1}{4}$. Since $\frac{1}{4}$ is not a value in the chart of common values, we have to use a calculator. Pressing $\cos^{-1} 4$ gives us 1.32

4 Write a trigonometric expression as an algebraic expression

If you are asked to find these inverse functions, the first step is always to denote the inverse function by θ. Let's look at an example.
<u>Ex 7</u> (Problem 61, page 453)
We are to express the expression $\sin(\sec^{-1} u)$ as an algebraic expression in u.
Step 1. Put $\theta = \sec^{-1} u$. The given expression is $\sin\theta$. We know that $\sec\theta = u$, or $\frac{1}{\cos\theta} = u$. Thus $\cos\theta = \frac{1}{u}$.
Step 2. Since we are given the cosine of θ, the range of θ is $0 \le \theta \le \pi$.
Step 3. Because $\sin\theta \ge 0$ in the above range, we have

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{u^2}}$$