Main points in Sections 7.3 and 7.4

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Contents

1 What to remember ?

The first thing to remember is the chart of common values :

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	1	0

The second is that only cosine and secant are even; the rest of trigonometric functions are odd. The last thing to remember is the two following identities :

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1, \\ \cot\theta &= \frac{1}{\tan\theta}. \end{aligned}$$

2 Solve an equation of a single trigonometric function

There are basically only 3 types of equations :

1) $\sin\theta = a$, where a is some specific number.
To solve this equation, we will find an angle θ_0 satisfying it. Then all solutions are $\theta = \theta_0 + 2k\pi$ and $\theta = \pi - \theta_0 + 2k\pi$ where k is an arbitrary integer.

2) $\cos\theta = a$, where a is some specific number.
To solve this equation, we will find an angle θ_0 satisfying it. Then all solutions are $\theta = \theta_0 + 2k\pi$ and $\theta = -\theta_0 + 2k\pi$ where k is an arbitrary integer.

3) $\tan\theta = a$, where a is some specific number.
To solve this equation, we will find an angle θ_0 satisfying it. Then all solutions are $\theta = \theta_0 + k\pi$ where k is an arbitrary integer.

Any equation involving $\cot\theta$, $\sec\theta$ or $\csc\theta$ can be transformed into one of these types.
Ex 1 (Problem 23, page 460)
The equation is $2\sin\theta + 1 = 0$. Subtracting 1 from both sides, we get $2\sin\theta = -1$. Dividing both sides by 2, we get

$$\sin\theta = -\frac{1}{2}$$

We do not see in the chart of common values any angle whose sine is $-\frac{1}{2}$. We only see $\sin\frac{\pi}{6} = \frac{1}{2}$. However, we know that sine is odd. Thus $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$. Thus $\theta_0 = -\frac{\pi}{6}$. Therefore all solutions of the given equation are

$$\theta = -\frac{\pi}{6} + 2k\pi \text{ and } \theta = \frac{7\pi}{6} + 2k\pi$$

By listing several terms, we see that only two values are in between 0 and 2π. They are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.
Ex 2 (Problem 24, page 460)
The equation is $\cos\theta + 1 = 0$.
Subtracting 1 from both sides, we get

$$\cos\theta = -1$$

From the unit circle, we observe that $\cos\pi = -1$. Thus $\theta_0 = \pi$. Therefore all solutions of the given equation are

$$\theta = -\pi + 2k\pi \text{ and } \theta = \pi + 2k\pi$$

By listing several terms, we see that only one value is in between 0 and 2π. It is π.
Ex 3 (Problem 17, page 460)
The equation is $\sin(3\theta) = -1$.
Put $t = 3\theta$. We see that the given equation is just

$$\sin t = -1$$

Looking at the unit circle, we see that $\sin\left(-\frac{\pi}{2}\right) = -1$. Thus $\theta_0 = -\frac{\pi}{2}$. Therefore,

$$t = -\frac{\pi}{2} + 2k\pi \text{ and } t = \frac{3\pi}{2} + 2k\pi$$

Because $t = 3\theta$, we divide t by 3 to get θ

$$\theta = -\frac{\pi}{6} + \frac{2k\pi}{3} \text{ and } \theta = \frac{\pi}{2} + \frac{2k\pi}{3}$$

By listing several terms, we see that only three values are in between 0 and 2π. They are $\frac{\pi}{2}$, $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.
<u>Ex 4</u> (Problem 18, page 460)
The equation is $\tan\frac{\theta}{2} = \sqrt{3}$.
Put $t = \frac{\theta}{2}$. We see that the given equation is just

$$\tan t = \sqrt{3}$$

Looking at the chart of common values, we see that $\tan\frac{\pi}{3} = \sqrt{3}$. Thus $\theta_0 = \frac{\pi}{3}$. Therefore,

$$t = \frac{\pi}{3} + k\pi$$

Since $t = \frac{\theta}{2}$, we multiply t by 2 to get θ

$$\theta = \frac{2\pi}{3} + 2k\pi$$

By listing several terms, we see that only one value is in between 0 and 2π. It is $\frac{2\pi}{3}$.
<u>Ex 5</u> (Problem 28, page 460)
The equation is $5\csc\theta - 3 = 2$.
Adding 3 to both sides, we get $5\csc\theta = 5$. Dividing both sides by 5, we get $\csc\theta = 1$. We know that $\csc\theta = \frac{1}{\sin\theta}$. Thus

$$\sin\theta = 1$$

Looking at the unit circle, we see that $\sin\frac{\pi}{2} = 1$. Thus $\theta_0 = \frac{\pi}{2}$. Therefore

$$\theta = \frac{\pi}{2} + 2k\pi$$

By listing several terms, we see that only one value is in between 0 and 2π. It is $\frac{\pi}{2}$.

3 Solve an equation involving more than one trigonometric function

The principle is to convert it into an equation of only one trigonometric function, either cosine, sine or tangent.

Ex 6 (Problem 63, page 461)
The equation is $\sin^2\theta - \cos^2\theta = 1 + \cos\theta$.
We can replace $\sin^2\theta$ by $1 - \cos^2\theta$ to get an equation of only cosine :

$$(1 - \cos^2\theta) - \cos^2\theta = 1 + \cos\theta,$$

which is equivalent to $1 - 2\cos^2\theta = 1 + \cos\theta$. Subtracting both sides by 1, we get $2\cos^2\theta = \cos\theta$. Adding $-2\cos^2\theta$ to both sides, we get $2\cos^2\theta + \cos\theta = 0$. We see that $\cos\theta$ is a common factor on the left. Thus we have $\cos\theta(2\cos\theta + 1) = 0$. From this we get two equations

$$\cos\theta = 0,$$

or

$$2\cos\theta + 1 = 0.$$

And now we already know how to solve each equation. The first equation gives

$$\theta = 2k\pi$$

The second gives

$$\theta = \frac{2\pi}{3} + 2k\pi \text{ and } \theta = -\frac{2\pi}{3} + 2k\pi$$

By listing several terms, we see that only three values are in between 0 and 2π. They are 0, $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.
Ex 7 (Problem 68, page 461)
The equation is $\cos\theta - \sin(-\theta) = 0$. We know that sine is an odd function, i.e. $\sin(-\theta) = -\sin(\theta)$. Thus $\cos\theta - \sin(-\theta) = \cos\theta - (-\sin\theta) = \cos\theta + \sin\theta$. Thus the given equation means $\cos\theta + \sin\theta = 0$. This equation involves sine and cosine. If we divide both sides by $\cos\theta$, we will see tangent :

$$\frac{\cos\theta + \sin\theta}{\cos\theta} = 0,$$

which is equivalent to $1 + \tan\theta = 0$, or

$$\tan\theta = -1$$

And we already know how to solve this equation : $\theta = -\frac{\pi}{4} + k\pi$.
By listing several terms, we see that only two values are in between 0 and 2π. They are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

4 Establishing identities

To establish an identity, we start from one side, either the left or right, trying to transform it to obtain the other side. The principle is to convert everything into

cosine and sine (or in some cases, tangent).
Ex 8 (Problem 22, page 470)

$$1 + \cot^2(-\theta) = \csc^2\theta$$

First we try to eliminate the minus sign to make things simpler. Since cotangent is odd, we have $\cot(-\theta) = \cot\theta$. Thus the left hand side is

$$1 + \cot^2(-\theta) = 1 + (-\cot\theta)^2 = 1 + \cot^2\theta$$

We should convert everything into sine and cosine. Since $\cot\theta = \frac{\cos\theta}{\sin\theta}$ and $\csc\theta = \frac{1}{\sin\theta}$, what we need to establish is

$$1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1}{\sin\theta}\right)^2$$

Now we start from the left hand side (LHS)

$$LHS = 1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2 = 1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} = \left(\frac{1}{\sin\theta}\right)^2 = RHS$$

Ex 9 (Problem 41, page 470)

$$1 - \frac{\cos^2\theta}{1 + \sin\theta} = \sin\theta$$

Now that everything is already sine or cosine, we start from the left hand side :

$$\begin{aligned} LHS &= 1 - \frac{\cos^2\theta}{1 + \sin\theta} \\ &= \frac{(1 + \sin\theta) - \cos^2\theta}{1 + \sin\theta} \\ &= \frac{(1 + \sin\theta) - (1 - \sin^2\theta)}{1 + \sin\theta} \\ &= \frac{1 + \sin\theta - 1 + \sin^2\theta}{1 + \sin\theta} \\ &= \frac{\sin\theta + \sin^2\theta}{1 + \sin\theta} \\ &= \frac{\sin\theta(1 + \sin\theta)}{1 + \sin\theta} \\ &= \sin\theta \\ &= RHS \end{aligned}$$