Main points in Sections 7.5 and 7.6

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1 What to remember in Section 7.5?

We need to memorize at least 3 following identities :

$$\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta} \tag{1}$$

$$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta} \tag{2}$$

$$\boxed{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}} \tag{3}$$

From them, we can deduce the formulas for $\cos(\alpha-\beta)$, $\sin(\alpha-\beta)$ and $\tan(\alpha-\beta)$. For example, to get $\cos(\alpha-\beta)$, we will replace β in (1) by $-\beta$. On the right hand side we will have $\cos(-\beta)$ and $\sin(-\beta)$. Because cosine is even and sine is odd, $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$. Then (1) gives us

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

As a consequence, for any two complementary angles, sine, cosine, tangent of one angle is equal to cosine, sine, cotangent of the other angle respectively. For example, because $30^0 + 60^0 = 90^0$, we have

$$\sin 30^0 = \cos 60^0, \quad \cos 30^0 = \sin 60^0, \quad \tan 30^0 = \cot 60^0.$$

2 Find trigonometric functions of a weird angle

When you are supposed to find the trigonometric functions of given an angle that is not 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ or $\frac{\pi}{2}$, you can follow the following steps :

1. Convert the given function into sine, cosine or tangent.
2. Express θ as a sum or difference of two familiar angles.
3. Apply Equations (1), (2) or (3).

Ex 1 (Problem 17, page 481)
We need to compute $\tan 15^0$. First, we see that the function is already tangent. Second, we see $15^0 = 45^0 - 30^0$. Thus we will use Eq. (3) with $\alpha = 45^0$ and $\beta = -30^0$. We have

$$\tan 15^0 = \tan(45^0 + (-30^0)) = \frac{\tan 45^0 + \tan(-30^0)}{1 - \tan 45^0 \tan(-30^0)}$$

We know that $\tan 45^0 = 1$. Because tangent is odd, we have $\tan(-30^0) = -\tan(30^0) = -\frac{\sqrt{3}}{3}$. Substituting these values into the above expression, we get $\tan 15^0 = 2 - \sqrt{3}$.

3 Some mixed problems in Section 7.5

Ex 2 (Problem 37, page 481)

$$\sin\alpha = \frac{5}{13}, \quad -\frac{3\pi}{2} < \alpha < -\pi; \quad \tan\beta = -\sqrt{3}, \ \frac{\pi}{2} < \beta < \pi.$$

(a) Compute $\sin(\alpha + \beta)$:
By Eq. (1), we have

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Thus we need to compute $\cos\alpha$, $\sin\beta$ and $\cos\beta$. We have

$$\cos^2\alpha = 1 - \sin^2\alpha = 1 - \left(\frac{5}{13}\right)^2 = \frac{144}{169}$$

Because $-\frac{3\pi}{2} < \alpha < -\pi$, it belongs to the third quadrant. Hence, $\cos\alpha < 0$ and

$$\cos\alpha = -\sqrt{\frac{144}{169}} = -\frac{12}{13}.$$

Next we will compute $\sin\beta$ and $\cos\beta$. We have

$$\cos^2\beta = \frac{1}{1+\tan^2\beta} = \frac{1}{1+(-\sqrt{3})^2} = \frac{1}{4}$$

Because $\frac{\pi}{2} < \beta < \pi$, it belongs to the second quadrant. Thus, $\cos\beta < 0$ and

$$\cos\beta = -\sqrt{\frac{1}{4}} = -\frac{1}{2}$$

We know $\tan\beta = \frac{\sin\beta}{\cos\beta}$. Thus,

$$\sin\beta = \cos\beta\tan\beta = \left(-\frac{1}{2}\right)(-\sqrt{3}) = \frac{\sqrt{3}}{2}$$

Now that we have $\sin\alpha$, $\cos\alpha$, $\sin\beta$ and $\cos\beta$, we have

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta = \frac{5}{13}\left(-\frac{1}{2}\right)+\left(-\frac{12}{13}\right)\frac{\sqrt{3}}{2} = \frac{-5-12\sqrt{3}}{26}$$

Doing similarly to Parts (b), (c) and (d), we have

(b)

$$\cos(\alpha+\beta) = \frac{12-5\sqrt{3}}{26}$$

(c)

$$\sin(\alpha-\beta) = \frac{-5+12\sqrt{3}}{26}$$

(d)

$$\tan(\alpha-\beta) = \frac{-240+169\sqrt{3}}{69}$$

4 What to remember in Section 7.6?

There are 3 important identities :

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta} \tag{4}$$

$$\boxed{\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta} \tag{5}$$

$$\boxed{\tan 2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}} \tag{6}$$

From these formulas, we can deduce other identities in the textbook. For example, from (5) we have $\cos 2\theta = 2\cos^2\theta - 1$. Adding 1 to both sides, we get $1+\cos 2\theta = 2\cos^2\theta$. Dividing both sides by 2, we get $\frac{1+\cos 2\theta}{2} = \cos^2\theta$, or

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

Putting $x = 2\theta$, we get

$$\cos^2\frac{x}{2} = \frac{1+\cos x}{2}$$

This is the half-angle formula for cosine.

Ex 3 (Problem 11, page 491)

$$\cos\theta = -\frac{\sqrt{6}}{3}, \quad \frac{\pi}{2} < \theta < \pi$$

(a) By Eq. (4) we have $\sin 2\theta = 2\sin\theta\cos\theta$. Thus, we need to compute $\sin\theta$ first. We have

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(-\frac{\sqrt{6}}{3}\right)^2 = 1 - \frac{6}{9} = \frac{3}{9} = \frac{1}{3}$$

Because $\frac{\pi}{2} < \theta < \pi$, it belongs to the second quadrant. Thus, $\sin\theta > 0$ and

$$\sin\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Hence

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{\sqrt{3}}{3}\left(-\frac{\sqrt{6}}{3}\right) = -2\frac{\sqrt{18}}{9} = -2\frac{3\sqrt{2}}{9} = -\frac{2\sqrt{2}}{3}$$

(b)

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{1}{\sqrt{3}}\right)^2 = 1 - \frac{2}{3} = \frac{1}{3}$$

(c)

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} = \frac{1+\frac{\sqrt{6}}{3}}{2} = \frac{\frac{3+\sqrt{6}}{3}}{2} = \frac{3-\sqrt{6}}{3}\times\frac{1}{2} = \frac{3+\sqrt{6}}{6}$$

Because $\frac{\pi}{2} < \theta < \pi$, we have $\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$. That means $\frac{\theta}{2}$ belongs to the first quadrant. Thus, $\sin\frac{\theta}{2} > 0$ and

$$\sin\frac{\theta}{2} = \sqrt{\frac{3+\sqrt{6}}{6}}$$

(d)

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} = \frac{1-\frac{\sqrt{6}}{3}}{2} = \frac{\frac{3-\sqrt{6}}{3}}{2} = \frac{3-\sqrt{6}}{3}\times\frac{1}{2} = \frac{3-\sqrt{6}}{6}$$

Because $\frac{\theta}{2}$ belongs to the first quadrant, $\cos\frac{\theta}{2} > 0$. Thus

$$\cos\frac{\theta}{2} = \sqrt{\frac{3-\sqrt{6}}{6}}$$