1. (20 points) The matrix

$$A = \begin{bmatrix} -5 & -4 \\ 12 & 9 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$. Find corresponding **eigenvectors** $\vec{v}_1$ and $\vec{v}_2$.

$$A - I = \begin{bmatrix} -6 & -4 \\ 12 & 8 \end{bmatrix} \xrightarrow[R_1 \to R_1/-6]{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$$

defective column $\rightsquigarrow$ $x_2 = s$, $x_1 = -\frac{2}{3}s$

$$\rightsquigarrow \vec{v}_1 = \begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} -8 & -4 \\ 12 & 6 \end{bmatrix} \xrightarrow[R_1 \to R_1/-8]{R_2 \to 2R_2 + 3R_1} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

defective column $\rightsquigarrow$ $x_2 = s$, $x_1 = -\frac{1}{2}s$

$$\rightsquigarrow \vec{v}_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

2. (30 points) Let $A$ be the $3 \times 3$ matrix with eigenvalues $\lambda_1 = 2$, $\lambda_2 = 0$, and $\lambda_3 = -1$; with corresponding eigenvectors $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v_3} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

(a) (10 points) What is the **kernel** (or nullspace) of $A$?

$$P = [\vec{v_1}\ \vec{v_2}\ \vec{v_3}] = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A = PDP^{-1} = \begin{bmatrix} -4/3 & -2 & 8/3 \\ 1/3 & 0 & -1/3 \\ 1/3 & -2 & 5/3 \end{bmatrix} \rightsquigarrow$$ then find nullspace by reducing $A$ into an echelon form

(b) (8 points) Find the **rank** of $A$.

the reduced echelon form of $A$ is

$$E = \begin{bmatrix} \boxed{1 \quad 0 \quad -1} \\ \boxed{0 \quad 1 \quad -1} \\ 0 \quad 0 \quad 0 \end{bmatrix} \rightsquigarrow \operatorname{rank}(A) = 2$$

(c) (12 points) What subspace of $\mathbb{R}^3$ is the **image** of $A$? Describe the image of $A$ in the form

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : ax_1 + bx_2 + cx_3 = 0 \right\}.$$

Note that the image of $A$ is the column space of $A$.

Column 1 and 2 of $E$ are pivot columns.

Thus, the column space of $A$ has a basis consisting of the first and second column of $A$.

$$\operatorname{col}(A) = \operatorname{span}\left\{ \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right), (-2, 0, -2) \right\}$$

To find $a, b, c$, we substitute $(x_1, x_2, x_3) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $(-2, 0, -2)$ and solve the system with unknowns $a, b, c$.

3. (35 points) Find a **particular solution** $y_p(t)$ for the nonhomogeneous differential equation

$$y''(t) - 4y'(t) = 9te^t.$$

Characteristic eq: $r^2 - 4r = 0 \left\langle \begin{array}{l} r = 0 \\ r = 4 \end{array} \right.$

Complementary function: $y_c(t) = c_1 e^{0t} + c_2 e^{4t} = c_1 + c_2 e^{4t}$.

Based on the righthand side, guess

$$y_p^*(t) = (at + b)e^t = ate^t + be^t$$

none appears in $y_c(t)$

$$\rightsquigarrow y_p(t) = y_p^*(t) = (at + b)e^t.$$

Want: $y_p'' - 4y_p' = 9te^t$

$$y_p' = ae^t + (at+b)e^t = (at + a + b)e^t$$

$$y_p'' = (at + 2a + b)e^t$$

$$y_p'' - 4y_p' = (-3at - 2a - 3b)e^t$$

we need: $\begin{cases} -3a = 9 \\ -2a - 3b = 0 \end{cases} \rightsquigarrow$ get $\begin{cases} a = -3 \\ b = 2 \end{cases}$

4. (40 points) The matrix

$$A = \begin{bmatrix} 5 & 3 & 6 \\ -10 & -8 & -10 \\ 4 & 4 & 5 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$, and $\lambda_3 = -3$; with corresponding eigenvectors $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v_3} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$. Consider the nonhomogeneous system of first-order DEs

$$\vec{x}'(t) = A\vec{x}(t) + \vec{b},$$

where $\vec{b} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$. A particular solution is $\vec{x_p}(t) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ (a constant solution).

(a) (25 points) Find the **general solution**.

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v_1} + c_2 e^{\lambda_2 t}\vec{v_2} + c_3 e^{\lambda_3 t}\vec{v_3} + \vec{x_p}$$

$$= c_1 e^{-t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t}\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(b) (15 points) Find the solution which satisfies the **initial condition** $\vec{x}(0) = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$.

$$\vec{x}(0) = c_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 - c_2 \\ c_2 - 2c_3 + 1 \\ -c_1 + c_3 - 1 \end{bmatrix} \text{ must be equal to } \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$$

$\rightsquigarrow$ get a system of 3 equations with 3 unknowns $c_1, c_2, c_3$.

5. 25 A certain piano plays the "A" above middle "C" on two strings. The piano tuner finds that one string sounds sharp, vibrating at 442 Hertz (442 cycles per second). The other string sounds flat, vibrating at 438 Hertz. Both strings are struck at the same time $t = 0$, producing sound waves $f_1(t)$ and $f_2(t)$ with initial conditions $f_1(0) = 1 = -f_2(0)$ and $f_1'(0) = f_2'(0) = 0$.

(a) (15 points) Write the two sound waves using angular frequencies $\omega_1 = \omega + \delta = 442/(2\pi)$ and $\omega_2 = \omega - \delta = 438/(2\pi)$.

$$f_1(t) = \cos(\omega_1 t) = \cos\left(\frac{442}{2\pi}t\right)$$

$$f_2(t) = -\cos(\omega_2 t) = -\cos\left(\frac{438}{2\pi}t\right)$$

(b) (10 points) Write the **sum** of the sound waves in the form of a product of sine functions.

$$f_1(t) + f_2(t) = \cos\left(\underbrace{\frac{442}{2\pi}}_{a}t\right) - \cos\left(\underbrace{\frac{438}{2\pi}}_{b}t\right)$$

$$= \cos(at) - \cos(bt)$$

$$= -2\sin\left(\frac{a-b}{2}t\right)\sin\left(\frac{a+b}{2}t\right).$$