Name____________________________________ November 8, 2016

Quiz 7

1. Evaluate the limit

$$\lim_{x \to 0} \frac{1 - \cos x}{x \sin x}$$

2. Find the points on the ellipse $x^2 + 9y^2 = 1$ that are farthest away from the point $(0, -8)$.

① $\lim_{x\to 0} \frac{\overbrace{1-\cos x}^{\frac{0}{0}}}{x \sin x} \underline{\underline{\text{L'Hospital}}} \lim_{x\to 0} \frac{\sin x}{\sin x + x\cos x}$ (still of the form $\frac{0}{0}$)

$\underline{\underline{\text{L'Hospital}}} \lim_{x\to 0} \frac{\cos x}{\cos x + (\cos x - x\sin x)}$

$= \frac{1}{1+(1-0)} = \boxed{\frac{1}{2}}$.

② The distance between a point (x,y) to $(0,-8)$ is

$$D = \sqrt{(x-0)^2+(y-(-8))^2} = \sqrt{x^2+(y+8)^2}$$

A point (x,y) belongs to the ellipse if $x^2+9y^2=1$. We obtain $x^2=1-9y^2$. Thus, $D=\sqrt{1-9y^2+(y+8)^2}=\sqrt{-8y^2+16y+65}$

We want to maximize D.

Note that the equation $x^2+9y^2=1$ gives us a constraint $y^2 \leq \frac{1}{9}$. In other words, $-\frac{1}{3} \leq y \leq \frac{1}{3}$.

It is more convenient to maximize

$$f(y) = D(y)^2 = -8y^2+16y+65 \text{ on } [-\tfrac{1}{3}, \tfrac{1}{3}].$$

We have $f'(y) = -16y+16 = 16(1-y) > 0$.

Thus, f is an increasing function. It attains maximum when $y=\frac{1}{3}$. Then $x^2 = 1-9y^2 = 1-9(\frac{1}{3})^2 = 0$. We get $\boxed{(0, \frac{1}{3})}$