# Research Statement

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My research interest is mainly in the theory of Partial Differential Equations and Analysis in general. The topics of my thesis are related to the Navier-Stokes Equations. In the following, I will briefly explain the results I obtained. Then I will describe a potential project and my other research plans for the future.

## 1 Summary of main results

In this section, I explain the main results of my thesis. They will appear in [11]. For $\Omega = \mathbb{R}^3$ or $\mathbb{R}^3_+$, we consider the initial boundary value problem for the Navier-Stokes equations in $\Omega \times (0, \infty)$

$$(\text{NSE})_\Omega : \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f & \text{in } \Omega \times (0,\infty), \\ \text{div } u = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{in } \partial\Omega \times (0,\infty), \\ u(\cdot, 0) = u_0 & \text{in } \Omega. \end{cases}$$

For convenience, we will denote the system for $\Omega = \mathbb{R}^3$ as (NSE), and $\Omega = \mathbb{R}^3_+$ as $(\text{NSE})_+$. We are interested in the case when the initial condition $u_0$ belongs to the space $L^3$, and the external force $f$ belongs to $L^{5/3}_{t,x}$. The norms in these spaces are invariant under the natural scalings $u_0(x) \to \lambda u_0(\lambda x)$ and $f(x,t) \to \lambda^3 f(\lambda x, \lambda^2 t)$. We will be considering the so-called mild solutions, which is a natural class of solutions for this problem. Fujita and Kato [3] essentially show that the problem is locally well-posed for any data $(u_0, f) \in L^3 \times L^{5/3}_{t,x}$, and globally well-posed for sufficiently small $(u_0, f) \in L^3 \times L^{5/3}_{t,x}$. The well-posedness for large data is still unknown. When $\Omega = \mathbb{R}^3$, this is more or less the millennium problem of fluid mechanics [2], modulo some technical differences between the formulations.

Let $\rho^\Omega_{\max}$ be the supremum of all $\rho > 0$ such that $(\text{NSE})_\Omega$ is globally well-posed for every $(u_0, f)$ with $\|(u_0,f)\|_{L^3 \times L^{5/3}_{t,x}} = \|u_0\|_{L^3} + \|f\|_{L^{5/3}_{t,x}} < \rho$. For convenience, we will denote the quantity for $\Omega = \mathbb{R}^3$ as $\rho_{\max}$, and $\Omega = \mathbb{R}^3_+$ as $\rho^+_{\max}$. Although it is not known whether $\rho^\Omega_{\max}$ is finite or infinite, we are interested in the hypothetical situation when $\rho^\Omega_{\max}$ is finite. In particular, we consider the following question:

(Q) If $\rho_{\max}^{\Omega}$ is finite, does there exist a data $(u_0, f) \in L^3 \times L^{5/3}_{t,x}$ with $\|(u_0, f)\| = \rho_{\max}^{\Omega}$, such that the solution $u$ of the system $(\text{NSE})_\Omega$ blows up in finite time?

We call such a data a minimal blowup data. This question was already addressed in [4, 5, 6, 12] for the case $\Omega = \mathbb{R}^3$ and $f = 0$. Their answers are affirmative in various settings of the initial condition, including $L^3$. Physical boundaries are known to play an important role in fluid flows. They often cause additional difficulties in the regularity theory (see e.g. [9, 16]), so it is natural to ask if the answer is still affirmative in the presence of boundaries. This motivates us to investigate the above problem in the half space $\mathbb{R}^3_+$, which is the simplest domain with boundary where the natural scalings still hold. Our results shed some light on how the boundary might influence the existence of minimal blowup data. We aim for the following results:

1) If $\rho_{\max} < \infty$ then there exists a minimal blowup data for (NSE).

2) $\rho^+_{\max} \leq \rho_{\max}$.

3) If $\rho^+_{\max} < \rho_{\max}$ then there exists a minimal blowup data for $(\text{NSE})_+$.

4) If $\rho^+_{\max} = \rho_{\max}$ then there does not exist a blowup data for $(\text{NSE})_+$.

These are the main goals of my thesis. The following result is proved in the thesis.

**Theorem 1** *The statements 1), 2), and 3) above hold true.*

The first statement says that the main result in [6] is still true in the case of nonzero right hand side, as long as $f \in L^{5/3}_{t,x}$. The results in [4, 5, 6, 12] are for the special case $f = 0$. The introduction of nontrivial $f$, beside its independent interest, enables us to overcome the difficulties related to stability issues when passing between two different domains.

When $\rho^+_{\max} < \rho_{\max}$, the boundary "helps" the blowup. In this case, all singularities (i.e. points $(x, t)$ around which $u$ is unbounded) stay close to the boundary. The main difficulty is to deal with the boundary regularity. This is handled by using the works of Seregin on the regularity of (suitable weak) solutions near a flat boundary (e.g. [13], [14]).

The case $\rho^+_{\max} = \rho_{\max}$ happens only when the singularities move away from the boundary. In this situation, the boundary seems to obstruct the existence of minimal blowup data. Other heuristics also seem to support the conjecture that when $\rho^+_{\max} = \rho_{\max}$ there does not exist a minimal blowup data. We plan to address this issue in the future.

The main new difficulty in the proofs (in comparison with previous works on minimal blowup solutions) is that the presence of the boundary complicates the partial regularity theory which is needed for the proofs. Our main tool in this respect is an adaptation of the results of [13] and [14], where the partial regularity theory near the boundary is developed.

## 2 A potential research project

We consider the initial value problem for the mollified Navier-Stokes equations

$$
(\text{NSE})_\varepsilon : \begin{cases} \partial_t u - \Delta u + (u * \eta_\varepsilon) \cdot \nabla u + \nabla p = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ \text{div } u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^3, \end{cases}
$$

where $\eta_\varepsilon = \varepsilon^{-3}\eta(x/\varepsilon)$ is a standard mollifier in $\mathbb{R}^3$. Denote by $(\text{NSE})_0$ the exact Navier-Stokes system. We consider the case when the initial condition $u_0$ belongs to $L^2 \cap L^\infty$. A natural class of solutions in this context is the so-called strong solutions (sometimes called mild solutions), defined in the space $L^\infty_{t,x}$.

In the pioneering work [7], Leray shows that $(\text{NSE})_0$ is locally well-posed, and $(\text{NSE})_\varepsilon$ is globally well-posed for each $\varepsilon > 0$ (see also [10]). The global well-posedness of $(\text{NSE})_0$ is still not known. Heuristically, the solution $u^\varepsilon$ of $(\text{NSE})_\varepsilon$ is more regular than the solution of $(\text{NSE})_0$ because the mollified nonlinear term in the equation is controlled by the linear terms. By a careful limit process, Leray constructs a global weak solution to $(\text{NSE})_0$ as a limit of $u^\varepsilon$. However, the uniqueness of Leray's weak solutions is not known due to possible lack of regularity of this class of solutions (see [2]).

Leray's construction is purely qualitative (i.e. using limit, compactness, etc.) Without a sufficiently strong a priori estimate, many regularity properties are lost in the limit $\varepsilon \to 0$. All a priori estimates that have been known so far can be traced back to the energy estimate, which is not strong enough to preserve the boundedness of solutions in the limit process (see [15, Sec. 3.4]). This motivates us to find a reasonable quantitative assumption on $u^\varepsilon$ that will indicate the existence of a global strong solution. Specifically, we consider the question:

> (Q) For $M > 0$, how large $\varepsilon$ can we take so that the following is true: "If $u^\varepsilon$ is bounded in $\mathbb{R}^3 \times (0,\infty)$ by $M$ then $(\text{NSE})_0$ has a global strong solution $u$ which is bounded in $\mathbb{R}^3 \times (0,\infty)$ by $2M$"?

This question was partially addressed by Li [8] from a numerical perspective. Li considers a discretized Navier-Stokes system in a polyhedron, in which the mesh size is analogous to $\varepsilon$. He essentially suggests that $\varepsilon \lesssim \exp(-M^{225})$. The mollified Navier-Stokes system seems to be a more natural model to study the above question because of the availability of scaling symmetry and the absence of boundaries. Like the exact Navier-Stokes system, $(\text{NSE})_\varepsilon$ also has a scaling symmetry:

$$
\begin{aligned}
u(x,t) &\to u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \\
p(x,t) &\to p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t), \\
\varepsilon &\to \varepsilon_\lambda = \lambda^{-1}\varepsilon.
\end{aligned}
$$

We are interested in obtaining a scale-invariant bound for $\varepsilon$. In the terminology of Caffarelli-Kohn-Nirenberg [1], each quantity in $(\text{NSE})_\varepsilon$ can be assigned a dimension. The bound $\|u^\varepsilon\|_{L^\infty} \leq M$ provides a natural length scale for the problem, which is $M^{-1}$. The number $\varepsilon$ also has dimension length, so the ideal bound for $\varepsilon$ would be $\varepsilon \lesssim M^{-1}$. Our goal is to investigate the following conjecture.

**Conjecture 2** *Let $M > 0$ and $u_0 \in L^2 \cap L^\infty$. There exists a constant $C > 0$ independent of $M$, possibly dependent on $\|u_0\|_{L^3}$, such that the following is true: Suppose for some $0 < \varepsilon \leq CM^{-1}$, the solution $u^\varepsilon$ of (NSE)$_\varepsilon$ is bounded by $M$. Then (NSE)$_0$ has a global strong solution which is bounded by $2M$.*

So far, we have obtained a partial result that $\varepsilon \leq F(M)$ where $F(M) \sim \exp(-M^3)$ as $M \to \infty$, improving the results of Li [8]. We believe that proving the result with $F(M) \sim M^{-\alpha}$, for some $\alpha > 0$, would already be significant.

The above problem has a variation as follows. Consider another approximate Navier-Stokes system

$$\text{(NSE)}_{\leq 1}: \begin{cases} \partial_t u - \Delta u + P_{\leq 1}(u \cdot \nabla u) + \nabla p = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \text{div } u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^3, \end{cases}$$

where $P_{\leq 1}$ is a Fourier multiplier that keeps all frequencies $\xi$ with $|\xi| \leq 1$ and suppresses all $\xi$ with $|\xi| \geq 2$. Consider the case when the initial condition belongs to the space $L^2$ and is compactly supported in the Fourier domain. Then (NSE)$_{\leq 1}$ is globally well-posed in the setting $L^\infty_t \dot{H}^s_x$ for any $s \geq 1/2$. Denote by $v$ its solution. A variant of Conjecture 2 might be the following.

**Conjecture 3** *Let $u_0 \in L^2$ with* supp $\hat{u}_0 \subset \{\xi : |\xi| \leq 2\}$. *For any $s > 1/2$ there exists $\delta = \delta(s, \|u_0\|_{\dot{H}^{1/2}})$ such that if $\|v(t)\|_{\dot{H}^s} \leq \delta$ for all $t > 0$ then* (NSE) *has a global solution $u$ belonging to $L^\infty_t \dot{H}^s_x$.*

So far we have obtained only partial results in this direction.

# 3 Other research plans

As a Ph.D student, I am at a very early stage of my career in research and education. I enjoy learning different aspects of mathematical analysis, and I am open to various research directions. In addition to the Navier-Stokes equations, I am also interested in other partial differential equations, such as the Euler equations, the Burgers equation, the harmonic map heatflow, and the Schrödinger equation. Other areas of mathematical analysis which I have strong interest in include harmonic analysis, geometric analysis, and stochastic partial differential equations, including particularly their applications in physics, biology and other disciplines. I plan to explore one or more of these fields in the future.

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