6651 Review Final I

1. Convert the following integral to an integral with variable θ using the substitution $3x = 5\sin\theta$.

$$\int_{5/6}^{5\sqrt{2}/6} \frac{x^2dx}{\sqrt{25-9x^2}}$$

$$dx = \frac{5}{3}\cos\theta\, d\theta$$

$$3\left(\frac{5\sqrt{2}}{6}\right) = 5\sin\theta \qquad \theta = \frac{\pi}{4}$$

$$3\left(\frac{5}{6}\right) = 5\sin\theta \qquad \theta = \frac{\pi}{6}$$

$$\frac{25}{27}\int_{\pi/6}^{\pi/4} \sin^2\theta\, d\theta$$

2. Find the following antiderivative showing all steps:

$$\int \frac{x^3}{\sqrt{x^2+9}}dx.$$

Let $u^2 = x^2+9$ $\quad u\,du = x\,dx$

$$\int \frac{(u^2-9)}{u}u\,du$$

$$= \frac{1}{3}u^3 - 9u + c = \frac{1}{3}(x^2+9)^{3/2} - 9(x^2+9)^{1/2} + c$$

3. Suppose $f(x)$ is a function such that $|f''(x)| \le 12$ for $2 \le x \le 8$. Find a bound for $|E_T|$ for the following integral when $n = 20$. Recall that $|E_T| \le \dfrac{K(b-a)^3}{12n^2}$.

$$\int_2^8 f(x)dx$$

$$\frac{12(8-2)^3}{12(20)^2} = \frac{27}{50}$$

b) Given that $|f''(x)| \le 20$ for $0 \le x \le 25$. Find the smallest value of n such that the error E_T made when evaluating the following integral using the trapezoid rule satisfies $|E_T| < 10^{-3}$.

$$\int_3^{15} f(x)dx$$

this is correct $\rightarrow$ $\dfrac{20(\cancel{25}\ 12)^3}{12n^2} < 10^{-3}$

$n_{min} = \cancel{5104}$

$n^2 > \cancel{(2880)10^3}$

$n > \cancel{1697.06}$

$n = \cancel{1698}$ $\leftarrow$ this is correct

4. If the following integral is convergent then evaluate it. If it is divergent, then explain why it is divergent.

$$\int_1^{\infty} \frac{xdx}{(x^2+8)^2}$$

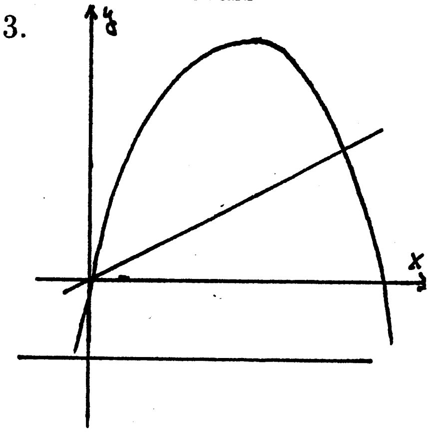
$$\int_1^b \frac{xdx}{(x^2+8)^2} = -\frac{1}{2}(x^2+8)^{-1}\Big|_1^b = \frac{1}{18} - \frac{1}{2(b^2+8)}$$

$$\lim_{n\to\infty}\left[\frac{1}{18} - \frac{1}{2(b^2+8)}\right] = \frac{1}{18}$$

Integral Converges

6652 Review Final II

1. The region bounded by the parabola $y = 6x - x^2$ and the line $y = x$ is covered by a lamina of density ρ. Find M_x, M_y, and the mass for this lamina. Find the center of mass of the lamina and the first moment of this lamina about the line $y = -3$.



See Attached Sheet

2. Solve the initial value problem

$$\frac{dy}{dx} - \frac{4}{x}y = x^6 \qquad \frac{dy}{dx} = \frac{4}{x}y + x^6 \qquad y(1) = \frac{7}{3}.$$

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = x^2$$

$$x^{-4}y = \frac{x^3}{3} + c$$

$$y = \frac{x^7}{3} + cx^4$$

$$y = \frac{x^7}{3} + 2x^4$$

3. Solve the following initial value problem. Solve for y.

$$\frac{dy}{dt} = 16 + 6y - y^2 \quad \text{and } y(0) = 6.$$

$$\frac{2+y}{8-y} = ce^{10t}$$

$$y = \frac{8ce^{10t} - 2}{1 + ce^{10t}}$$

$$y = \frac{32e^{10t} - 2}{1 + 4e^{10t}}$$

$c = 4$

4. A very large tank contains 400 gallons of water with 50 lbs of salt dissolved in the water. Brine containing 3/2 lbs of salt per gallon is pumped into the tank at the rate of 8 gallons per minute. The mixture is pumped out of the tank at the slower rate of 6 gallons per minute. Find an expression for the amount of salt in the tank at time t.

See Attached Sheet

1. $\text{Mass} = \rho \int_0^5 (6x - x^2 - x)\,dx = \frac{5^3}{6}\rho = \frac{125\rho}{6}$

$$M_x = \frac{\rho}{2}\int_0^5 \left[(6x - x^2)^2 - x^2\right] dx$$

$$= \frac{\rho}{2}\ \frac{35x^3}{3} - 3x^4 + \frac{x^5}{5}\Big|_0^5 = \frac{5^4}{6}\rho = \frac{625}{6}\rho$$

$$M_y = \rho \int_0^5 x\left[(6x - x^2) - x\right] dx$$

$$\rho\ \frac{5x^3}{3} - \frac{x^4}{4}\Big|_0^5 = \rho\,\frac{5^4}{12} = \frac{625\rho}{12}$$

$$(\bar{x}, \bar{y}) = \left(\frac{5}{2}, 5\right)$$

Moment About the line $y = -3$ is

$$M_x + 3(\text{Mass})$$

$$= \frac{500}{3}\rho$$

4. Rate in $= \left(\frac{3}{2}\ \text{lb}/\text{gal}\right)\left(8\ \text{gal}/\text{min}\right) = 12\ \text{lbs}/\text{min}$

Rate out $= \left(6\ \text{gal}/\text{min}\right)\left(\frac{y}{400+2t}\ \text{lbs}/\text{gal}\right) = \frac{3y}{200+t}$

$$\frac{dy}{dt} = 12 - \frac{3}{200+t}\,y$$

$$(200+t)^3 y = 3(200+t)^4 + c$$

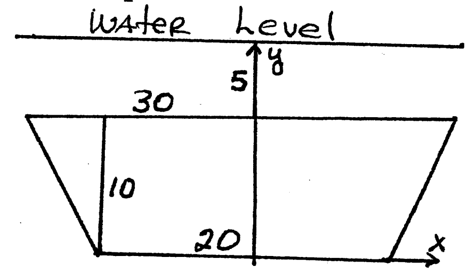
$$y = 3(200+t) + c(200+t)^{-3}$$

$$c = -550(200)^3$$

$$y = 3(200+t) - 550(200)^3(200+t)^{-3}$$

6653 Review Final III

1. At the end of a pool there is a large glass plate in the shape of a trapezoid with equal sides. The trapezoid is 30 ft long at the top and 20 feet long at the bottom. The plate is 10 feet high and the top of the plate is 5 feet below the water level. If the pool is full of water, find the total force on the glass plate.



$$\Delta F = (15-y)(2x)(\Delta y)\,\rho$$

$$\text{Force} = \int_0^{10} \rho\,(15-y)(y+20)\,dy$$

$$= \frac{29}{12}(10^3)\,\rho = \frac{7250}{3}\rho$$

$$y = 2(x-10)$$

2. Find the area of the region which is outside the circle $r = 6\cos\theta$ but inside the cardioid $r = 2 + 2\cos\theta$.

See Attached Sheet

3. Find the area of the region which is inside the cardioid $r = \sin\theta - 1$ which is also above the line $\theta = \pi/4$ and in the first quadrant.

See Attached Sheet

4. Evaluate the following indefinite integral. Show all steps.

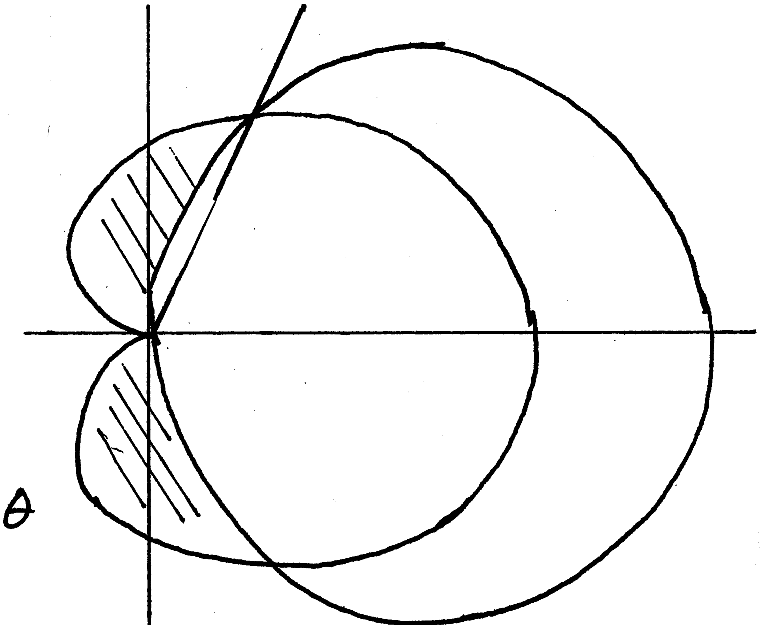
$$\int \frac{x^2 + 13x + 4}{(x^2+4)(x+3)}\,dx$$

$$\int \left[\frac{3x+4}{x^2+4} - \frac{2}{x+3}\right] dx =$$

$$\frac{3}{2}\ln(x^2+4) + 2\arctan\left(\frac{x}{2}\right) - 2\ln|x+3| + C$$

2. $\displaystyle\int_{\frac{\pi}{3}}^{\pi} (2+2\cos\theta)^2 d\theta$

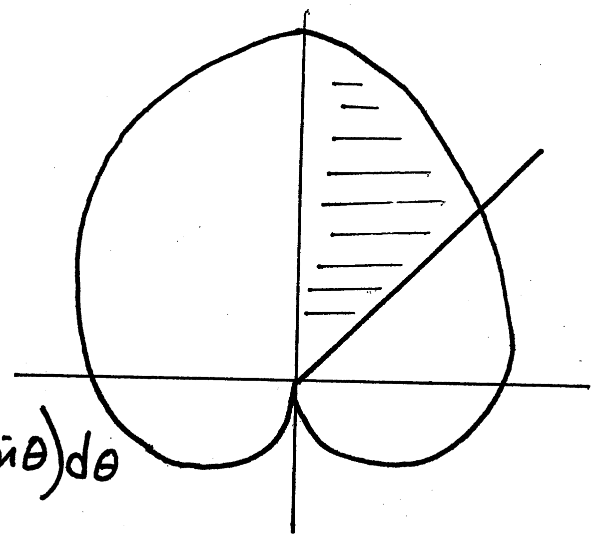
$\displaystyle - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (6\cos\theta)^2 d\theta$



$\displaystyle = \int_{\pi/3}^{\pi} (6+8\cos\theta+2\cos 2\theta)d\theta$

$\displaystyle - \int_{\pi/3}^{\pi/2} (18+18\cos 2\theta)d\theta$

$= \pi$



3. $\displaystyle \frac{1}{2}\int_{5\pi/4}^{3\pi/2} (\sin\theta-1)^2 d\theta$

$\displaystyle = \frac{1}{2}\int_{5\pi/4}^{3\pi/2} \left(\frac{3}{2}-\frac{1}{2}\cos 2\theta-2\sin\theta\right)d\theta$

$\displaystyle = \frac{3\theta}{4}-\frac{1}{8}\sin 2\theta+\cos\theta \Big|_{5\pi/4}^{3\pi/2}$

$\displaystyle = \frac{3\pi}{16}+\frac{1}{8}+\frac{\sqrt{2}}{2}$

6654 Review Final IV

1. Show that the following alternating series is convergent.

$$\sum_{n=1}^{\infty}(-1)^n \frac{8n+5}{n(2n+1)}$$

See Attached Sheet

2. First, find values of p and c such that $\frac{8n+7}{n(n^2+1)} \leq \frac{c}{n^p}$. Show that the series is convergent using the comparison theorem.

$$\sum_{n=1}^{\infty} \frac{8n+7}{n(n^2+1)}$$

See Attached Sheet

3. For what values of x is the following power series convergent? divergent?

$$\sum_{n=0}^{\infty} \frac{2^n(n+2)x^n}{5^n(3n+5)}$$

See Attached Sheet

4. The Taylor polynomial $T_5(x)$ for $(4+x)^{3/2}$ is

$$(4+x)^{3/2} \approx 8+3x+\frac{3}{16}x^2-\frac{1}{128}x^3+\frac{3}{4096}x^4-\frac{3}{32768}x^5$$

Use this polynomial to find Taylor polynomials for $(4+x)^{1/2}$ and $(4+x)^{5/2}$.

See Attached Sheet

1. the Alternating Series theorem says:
If $\lim_{n\to\infty} \frac{8n+5}{n(2n+1)} = 0$ And $\frac{8n+13}{(n+1)(2n+3)} < \frac{8n+5}{n(2n+1)}$,
then $\sum_{n=1}^{\infty} (-1)^n \frac{8n+5}{n(2n+1)}$ converges. Clearly, $\lim_{n\to\infty} \frac{8+5/n}{2n+1} = 0$.
$n(2n+1)(8n+13) < (8n+5)(n+1)(2n+3)$ is the same as
$16n^3+34n^2+13n < 16n^3+50n^2+49n+15$ which is
true if $0 < 16n^2+36n+15$ which is clearly true.
We conclude from those three statements taken together that $\sum_{n=1}^{\infty} (-1)^n \frac{8n+5}{n(2n+1)}$ converges.

2. $\frac{8n+7}{n(n^2+1)} = \frac{8+7/n}{n^2+1} \simeq \frac{8}{n^2}$. Now $\frac{8n+7}{n(n^2+1)} < \frac{9}{n^2}$
is true if $8n^2+7n < 9n^2+9$ which is true if
$7n < n^2+9$ which is true for $n=1$ and $n \geq 6$.
the series $\sum_{n=1}^{\infty} \frac{9}{n^2}$ converges since it is a
p-series with $p=2$ and $c=9$. the comparison
test theorem says: If $\sum_{n=1}^{\infty} \frac{9}{n^2}$ converges and
if $\frac{8n+7}{n(n^2+1)} < \frac{9}{n^2}$, then $\sum_{n=1}^{\infty} \frac{8n+7}{n(n^2+1)}$ converges.
We conclude that $\sum_{n=1}^{\infty} \frac{8n+7}{n(n^2+1)}$ converges.

3. the Ratio is

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+3)X^{n+1}}{5^{n+1}(3n+8)} \cdot \frac{5^n(3n+5)}{2^n(n+2)X^n}$$

$$= \frac{2(n+3)(3n+5)}{5(3n+8)(n+2)} X$$

$$L = \lim_{n\to\infty} \frac{2(1+3/n)(3+5/n)|X|}{5(3+8/n)(1+1/n)} = \frac{6|X|}{15} = \frac{2|X|}{5}$$

the Ratio test theorem says:

If $|X| < \frac{5}{2}$, then $\sum_{n=0}^{\infty} \frac{2^n(n+2)X^n}{5^n(3n+5)}$ converges.

If $|X| > \frac{5}{2}$, then $\sum_{n=0}^{\infty} \frac{2^n(n+2)X^n}{5^n(3n+5)}$ diverges.

4. Differentiating both sides

$$\frac{3}{2}(4+x)^{1/2} \sim 3 + \frac{3}{16}(2x) - \frac{1}{128}(3X^2) + \frac{3}{4096}(4X^3) - \frac{3}{32768}(5X^4)$$

$$(4+x)^{1/2} \simeq 2 + \frac{1}{4}x - \frac{1}{64}X^2 + \frac{1}{512}X^3 - \frac{5}{16384}X^4$$

4. Cont.

Integrating both sides

$$\frac{(4+x)^{5/2}}{5/2} \sim C + 8x + 3\frac{x^2}{2} + \frac{3}{16}\frac{x^3}{3} - \frac{1}{128}\frac{x^4}{4} + \frac{3}{4096}\frac{x^5}{5} - \frac{3}{32768}\frac{x^6}{6}$$

$$(4+x)^{5/2} \sim \frac{5}{2}C + 20x + \frac{15}{4}x^2 + \frac{5}{32}x^3 - \frac{5}{1024}x^4 + \frac{3}{8192}x^5 - \frac{5}{131072}x^6$$

$x = 0$ gives $\frac{5}{2}C = 32$

$$(4+x)^{5/2} = 32 + 20x + \frac{15}{4}x^2 + \frac{5}{32}x^3 - \frac{5}{1024}x^4 + \frac{3}{8192}x^5 - \frac{5}{131072}x^6$$

6655 Review Final V

1. Consider the graph of the following vector function. Find the length of the arc of this curve from $(0,3,4)$ to $(16,27,10)$

$$\vec{r}(t) = [(4\sqrt{2})t^{3/2}]\vec{i} + (6t^2+3)\vec{j} + (3t+4)\vec{k}.$$

$$\vec{r}'(t) = (6\sqrt{2}\,t^{1/2})\vec{i} + (12t)\vec{j} + 3\vec{k}$$

$$\|\vec{r}'(t)\|^2 = 72t + 144t^2 + 9$$

$$\text{Arc length} = \int_0^2 3(4t+1)\,dt = 30$$

2. If $N=8$ for what values of x is $\left|\arctan x - \sum_{k=0}^{N}(-1)^k \frac{x^{2k+1}}{2k+1}\right| < 10^{-6}$.

$$\left|\sum_{k=N+1}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}\right| < \frac{|x|^{2N+3}}{2N+3} \quad \text{for } |x|<1$$

We Need $\frac{|x|^{19}}{19} < 10^{-6}$ $\qquad |x| < 0.5643$

3. If $|x| \le 0.8$ find the smallest value of N such that Do NOT Round up

Find N such that $\left|\sin x - \sum_{k=0}^{N}(-1)^k \frac{x^{2k+1}}{(2k+1)!}\right| < 10^{-8}$.

$$\frac{(0.8)^{2N+3}}{(2N+3)!} < 10^{-8}$$

$$\frac{(0.8)^9}{9!} = 3.7\times 10^{-7} > 10^{-8}$$

$N = 4$

$$\frac{(0.8)^{11}}{11!} = 2.2\times 10^{-9} < 10^{-8}$$

4. Given $\vec{v} = 3i - 5j + 2k$ and $\vec{w} = 4\vec{i} + 3\vec{j} - \vec{k}$ find the angle between $\vec{v}$ and $\vec{w}$. Find $\text{Proj}_{\vec{w}}\vec{v}$.

$$\vec{v}\cdot\vec{w} = -5$$

$$\|\vec{v}\|^2 = 38 \quad \|\vec{w}\|^2 = 26$$

$$\cos\theta = \frac{-5}{\sqrt{26}\,\sqrt{38}}$$

$$\theta = 1.7305$$

$$\text{Proj}_{\vec{w}}\vec{v} = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}\vec{w}$$

$$= \frac{-5}{26}(4i+3j-k)$$

6656 Review Final VI

1. Evaluate the following indefinite integral clearly showing the u substitution

$u = 3x^2 + 2\sin 3x$ $\qquad \int (3x^2 + 2\sin 3x)^{3/5}(x + \cos 3x)dx$

$$\int u^{3/5}\frac{du}{6} = \frac{5}{48}u^{8/5} + C = \frac{5}{48}\left[3x^2 + 2\sin 3x\right]^{8/5} + C$$

2. Find the vector function $\vec{R}(s)$ whose graph is the tangent line at the point $(16, 20, 7)$ to the curve which is the graph of $\vec{r}(t) = (t^3 + 4t)\vec{\imath} + (3t^2 + 8)\vec{\jmath} + (t^2 + 4t - 5)\vec{k}$.

$$\vec{r}'(t) = (3t^2 + 4)\vec{i} + (6t)\vec{j} + (2t + 4)\vec{k}$$

$$\vec{r}'(2) = 16\vec{i} + 12\vec{j} + 8\vec{k}$$

$$\vec{R}(s) = 16\vec{i} + 12\vec{j} + 8\vec{k} + s\left[16\vec{i} + 12\vec{j} + 8\vec{k}\right]$$

3. Consider the following geometric series. What number term is 59049/16?

$$64 + 96 + 144 + 216 + \ldots + (59049/16)$$

$a_1 = 64$

$r = \frac{3}{2}$

$$64\left(\frac{3}{2}\right)^{n-1} = \frac{59049}{16}$$

$$\left(\frac{3}{2}\right)^{n-1} = \frac{59049}{1024} = \left(\frac{3}{2}\right)^{10}$$

$n = 11$

4. Find the area of the region which is inside the cardioid $r = -2\cos\theta - 2$ and the circle $r = -6\cos\theta$ and also in the second quadrant.

See Attached Sheet

5. The following are the rectangular coordinates of some points. Find four sets of polar coordinates for these points.

(a) $(-4\sqrt{3}, 4)$ (b) $(-6, -6)$

a) $\left(8, \frac{5\pi}{6}\right)$ $\quad \left(-8, \frac{11\pi}{6}\right)$ $\quad \left(8, -\frac{7\pi}{6}\right)$ $\quad \left(-8, -\frac{\pi}{6}\right)$

b) $\left(6\sqrt{2}, \frac{5\pi}{4}\right)$ $\quad \left(-6\sqrt{2}, \frac{\pi}{4}\right)$ $\quad \left(6\sqrt{2}, -\frac{3\pi}{4}\right)$ $\quad \left(-6\sqrt{2}, -\frac{7\pi}{4}\right)$

6. Find the sum $-13 - 7 - 1 + 5 + 11 \ldots + 1199$.

$a_1 = -13 \qquad d = 6$

$1199 = -13 + (n-1)(6)$

$n = 203$

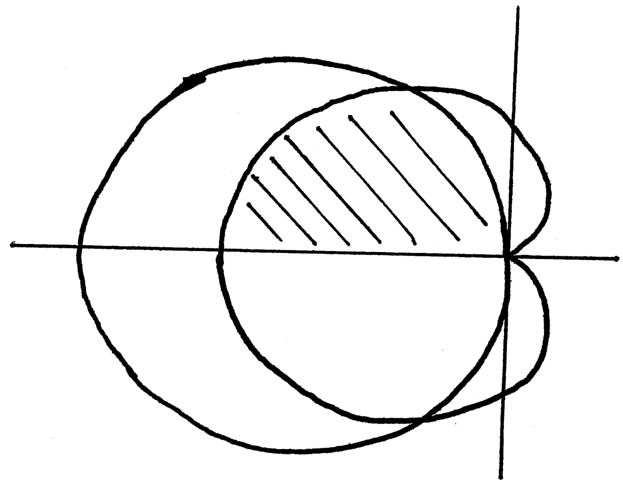
$$S_{203} = \frac{203}{2}\left[-13 + 1199\right]$$

$$= 120,379$$

4.

$$-6\cos\theta = -2\cos\theta - 2$$

$$\theta = \frac{\pi}{3} \quad \frac{5\pi}{3}$$



$$\text{Area} = \frac{1}{2}\int_{\frac{5\pi}{3}}^{2\pi} (-2\cos\theta - 2)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{2}}^{2\pi/3} (-6\cos\theta)^2 d\theta$$

$$= \frac{5\pi}{2}$$