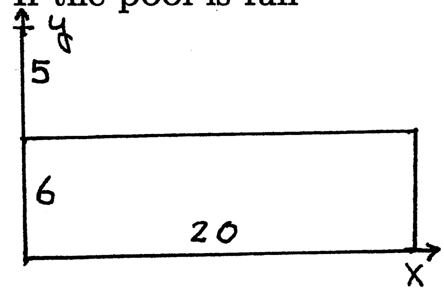
6543 Hydrostatic Force

1. A glass window at the end of a pool is in the shape of a rectangle. The rectangle is 20 feet long and 6 feet high. The top of the rectangle is 5 feet below the surface of the water. The pool is 12 feet deep. If the pool is full of water, find the hydrostatic force on the window.



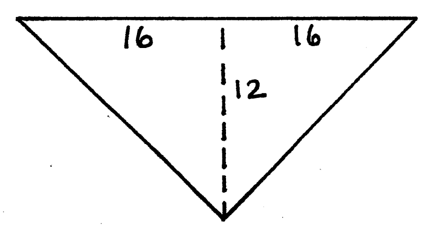
Hydrostatic force = weight of fluid above the element. All parts of the element must be at the same depth. Hydrostatic force = depth times area of element times density.

$$\Delta F = (11-y)\, x\, \Delta y\, \rho$$

$$\text{Hydrostatic force} = (62.5)\int_0^6 (11-y)(20)\,dy$$

$$= 1250\left[11y - \frac{y^2}{2}\Big|_0^6\right] = 60{,}000 \text{ pounds}$$

2. The end of a certain tank is in the shape of an isosceles triangle. The tip of the triangle is down. The top of the triangle is 32 feet across. The tank is 12 feet deep at the tip of the triangle. The tank is 40 feet long. If the tank is full of water, find the hydrostatic force on the end of the tank. Assume water weighs 62.5 lbs/ft^3.



$$\Delta F = (12-y)(2x)\,\Delta y\,\rho$$

$$y = \frac{3}{4}x$$

$$x = \frac{4}{3}y$$

Hydrostatic force

$$= (62.5)(2)\int_0^{12} (12-y)\left(\frac{4}{3}y\right)dy$$

$$= \frac{500}{3}\left[6y^2 - \frac{y^3}{3}\Big|_0^{12}\right] = 48{,}000$$

6544 A Logistic Equation

1. Find the area of the surface generated when the curve $y = x^{3/2} - (1/3)x^{1/2}$ from $(1, 2/3)$ to $(9, 26)$ is revolved about the x axis.

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{3}{2}x^{1/2} + \frac{1}{6}x^{-1/2}\right)^2$$

$$\left(\frac{3}{2}x^{1/2} + \frac{1}{6}x^{-1/2}\right)\left(x^{3/2} - \frac{1}{3}x^{1/2}\right) = \frac{3}{2}x^2 - \frac{1}{3}x - \frac{1}{18}$$

$$2\pi\int_1^9 \left(\frac{3}{2}x^2 - \frac{1}{3}x - \frac{1}{18}\right)dx = \frac{6304}{9}\pi$$

2. Find the general solution of the following differential equation. Also find the particular solution which satisfies the initial condition $y(0) = 2$ and the particular solution which satisfies the initial condition $y(0) = 9$.

$$\frac{dy}{dt} = y(y-6).$$

$$\left(\frac{1}{y-6} - \frac{1}{y}\right)dy = 6\,dt$$

$$\ln|y-6| - \ln|y| = 6t + \ln C$$

$$\ln\left|\frac{y-6}{y}\right| = \ln\left[Ce^{6t}\right]$$

$$\left|\frac{y-6}{y}\right| = Ce^{6t} \qquad C>0$$

this is equivalent to

$$\frac{y-6}{y} = Ce^{6t} \qquad C \neq 0$$

$$y = \frac{6}{1 - Ce^{6t}}$$

Note $y = 6$, $C = 0$ is Also A Solution

$y(0) = 2$ gives $C = -2$

$$y(t) = \frac{6}{1 + 2e^{6t}}$$

$y(0) = 9$ gives $C = \frac{1}{3}$

$$y(t) = \frac{18}{3 - e^{6t}}$$

6545 Miscellaneous

1. Find the following antiderivative or indefinite integral

$$\int \cos(10x)\sin(6x)dx.$$

$\int \cos(10x)\sin(6x)\,dx$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$= \frac{1}{2}\int [\sin(16x) + \sin(-4x)]\,dx$

$= -\frac{1}{32}\cos(16x) + \frac{1}{8}\cos(4x) + C$

$= \frac{3}{32}\cos(10x)\cos(6x) + \frac{5}{32}\sin(10x)\sin(6x) + C$

2. Use the substitution $u = 3x + 16$ to convert the following definite integral into a definite integral of equal value with u as the variable of integration.

$$\int_{11}^{35} \frac{x\,dx}{\sqrt{3x+16}}$$

$u = 3x + 16 \qquad du = 3\,dx$

when $x = 11 \qquad u = 49$

when $x = 35 \qquad u = 121$

$\int_{49}^{121} \frac{\frac{u-16}{3}\,\frac{du}{3}}{u^{1/2}} = \frac{1}{9}\int_{49}^{121} [u^{1/2} - 16u^{-1/2}]\,du$

6537 Practice Gateway Exam

Evaluate the following integrals. You do not need to simplify your answers. Only correct answers will receive credit. No partial credit will be given.

Given: $2\cos^2 x = 1 + \cos 2x$, $2\sin^2 x = 1 - \cos 2x$, $\sin^2 x + \cos^2 x = 1$

1. Convert the following integral to an integral with variable θ and with correct limits using the substitution $5x = 4\sin\theta$:

$5\,dx = 4\cos\theta\,d\theta$

$$\int_0^{2\sqrt{3}/5} \frac{x^2}{\sqrt{16 - 25x^2}}\,dx.$$

$$\int_0^{\pi/3} \frac{\frac{16}{25}\sin^2\theta \; \frac{4}{5}\cos\theta\,d\theta}{\sqrt{16 - 16\sin^2\theta}}$$

Ans: $\frac{16}{125}\int_0^{\pi/3} \sin^2\theta\,d\theta$

2. Convert the following integral to an integral with variable θ and with correct limits using the substitution $x = 4\tan\theta$:

$$\int_4^{4\sqrt{3}} \frac{x^2\,dx}{\sqrt{16 + x^2}}.$$

$dx = 4\sec^2\theta\,d\theta$

when $x = 4$, $\tan\theta = 1$, $\theta = \frac{\pi}{4}$

when $x = 4\sqrt{3}$, $\tan\theta = \sqrt{3}$, $\theta = \frac{\pi}{3}$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{(16\tan^2\theta)(4\sec^2\theta)\,d\theta}{\sqrt{16 + 16\tan^2\theta}}$$

Ans: $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 16\tan^2\theta\sec\theta\,d\theta$

3. Evaluate $\int \frac{x^3 dx}{\sqrt{1+3x^2}}$

$u = 1 + 3x^2 \qquad \frac{u-1}{3} = x^2$

$du = 6x\,dx$

$$\int \frac{\frac{u-1}{3}\ \frac{du}{6}}{u^{1/2}} = \frac{1}{18}\int u^{1/2} - u^{-1/2}\,du$$

$$= \frac{u^{3/2}}{27} - \frac{u^{1/2}}{9} + C$$

Ans: $\frac{1}{27}(1+3x^2)^{3/2} - \frac{1}{9}(1+3x^2)^{1/2} + C$

4. Evaluate $\int \sin^2(5x)dx$

$$= \frac{1}{2}\int (1 - \cos(10x))\,dx$$

Ans: $\frac{x}{2} - \frac{1}{20}\sin(10x) + C$

5. Evaluate $\int \sin^2(4x)\cos^3(4x)dx$

$$\frac{1}{4}\int [\sin^2(4x)]\,[1 - \sin^2(4x)]\,[\cos(4x)]\,4dx$$

$$= \frac{1}{12}\sin^3(4x) - \frac{1}{20}\sin^5(4x) + C$$

Ans:

6. Evaluate $\int \frac{6x+20}{25+9x^2}dx$ $= \int \frac{6x\,dx}{25+9x^2} + \int \frac{20dx}{25+9x^2}$

Consider

$\int \frac{6x\,dx}{25+9x^2}$ Let $u = 25+9x^2$ $x\,dx = \frac{du}{18}$

$\frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln(25+9x^2) + c$

Consider $\int \frac{20\,dx}{25+9x^2}$ Let $5u = 3x$ $dx = \frac{5}{3}du$

$\frac{4}{3}\int \frac{du}{1+u^2} = \frac{4}{3}\arctan u + c$

$\frac{1}{3}\ln(25+9x^2) + \frac{4}{3}\arctan\left(\frac{3x}{5}\right) + c$

Ans:____________________

7. Evaluate $\int \frac{(72x-20)dx}{(3x^2+5)(x+6)}$

$\int \left[\frac{12x}{3x^2+5} - \frac{4}{x+6}\right]dx$

Ans:____________________

$= 2\ln(3x^2+5) - 4\ln|x+6| + c$

8. Evaluate $\int x \cos 4x dx$

$$\frac{1}{4} x \sin(4x) + \frac{1}{16} \cos(4x) + C$$

9. Evaluate $\int \sec^4(5x)\ dx$

$$\frac{1}{5} \int \left[1 + \tan^2(5x)\right] \sec^2(5x)\,(5dx)$$

$$= \frac{1}{5} \tan(5x) + \frac{1}{15} \tan^3(5x) + C$$

10. Evaluate $\int \frac{[x - \sin(4x)]dx}{[2x^2 + \cos(4x)]^{4/3}}$

Let $u = 2x^2 + \cos(4x)$

$du = [4x - 4\sin(4x)]dx$

$= 4[x - \sin(4x)]dx$

$$\int u^{-4/3} \frac{du}{4} = \frac{1}{4} \frac{u^{-1/3}}{-1/3} + C$$

$$-\frac{3}{4}\left[2x^2 + \cos(4x)\right]^{-1/3} + C$$