6553 Review: Substitution

1. Find the following antiderivative using a u-substitution. Show all steps.

$$\int (4x^2 + 16x + 11)^{3/5}(x+2)\,dx.$$

Let $u = 4x^2 + 16x + 11$

$$du = (8x+16)dx = 8(x+2)dx$$

$$\frac{1}{8}\int u^{3/5}\,du = \frac{1}{8}\frac{u^{8/5}}{8/5} + c = \frac{5}{64}u^{8/5} + c$$

$$= \frac{5}{64}\left[4x^2 + 16x + 11\right]^{8/5} + c$$

2. Convert the following definite integral into another definite integral with variable of integration θ using the substitution $3x = 5\sin\theta$. Show all work.

$$dx = \frac{5}{3}\cos\theta\,d\theta \qquad \int_{5/6}^{5\sqrt{3}/6} \frac{x^2\,dx}{(25-9x^2)^{3/2}}$$

when $x = \frac{5}{6}$ $\quad \sin\theta = \frac{1}{2}$ $\quad \theta = \frac{\pi}{6}$

when $x = \frac{5\sqrt{3}}{6}$ $\quad \sin\theta = \frac{\sqrt{3}}{2}$ $\quad \theta = \frac{\pi}{3}$

$$\frac{1}{27}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2\theta\,d\theta$$

3. Evaluate the following integral:

$$\int \frac{64x+6}{(x^2+16)(x-9)}\,dx.$$

$$\int \frac{-6x+10}{x^2+16} + \frac{6}{x-9}\,dx$$

$$= -3\ln(x^2+16) + \frac{5}{2}\arctan\left(\frac{x}{4}\right) + 6\ln|x-9| + c$$

6554 Review: Moments

1. Calculate the first moments M_x and M_y for the lamina of density ρ which covers the region bounded by the parabola $y = 5x - x^2$ and the line $y = x$. Find the mass of the lamina. Also find the first moment about the line $x = -3$.

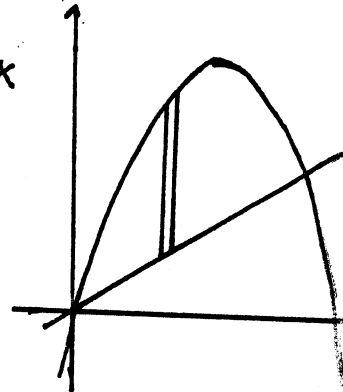$$\Delta M_x = \frac{\rho}{2} y_1^2 \Delta x - \frac{\rho}{2} y_2^2 \Delta x = \frac{\rho}{2}\left[(5x-x^2)^2 - x^2\right]\Delta x$$

$$M_x = \rho \int_0^4 \left(12x^2 - 5x^3 + \tfrac{1}{2}x^4\right)dx = \frac{192}{5}\rho$$

$$\Delta M_y = \rho x y_1 \Delta x - \rho x y_2 \Delta x = \rho x\left[5x - x^2 - x\right]\Delta x$$

$$M_y = \int_0^4 x(5x - x^2 - x)\,dx = \frac{64}{3}\rho$$

$$MASS = \rho(AREA) = \frac{32}{3}\rho$$

$$Moment = 3(MASS) + \frac{64}{3}\rho = \frac{160}{3}\rho$$

2. The actual error E_T made when using the trapezoid rule to approximate an integral satisfies $|E_T| \le \frac{K(b-a)^3}{12n^2}$. Find a bound on the error when approximating the integral $\int_1^5 (5x + 8x^3 - x^4)dx$ for the trapezoid rule using $n = 200$ subintervals.

$$f''(x) = 48x - 12x^2$$
$$f'''(x) = 48 - 24x$$
$$f'''(1) = 24 \quad local\ min$$
$$f'''(5) = -72 \quad local\ min$$
$$f'''(2) = -24 \quad local\ max$$

$$f''(1) = 36, \quad f''(2) = 48, \quad f''(5) = -$$

$$|f''(x)| \le 60 \quad for\ 1 \le x \le 5$$
$$K = 60$$

$$|E_T| \le \frac{60(5-1)^3}{12(200)^2}$$

$$= \frac{1}{125} = 0.008$$

3. Suppose $f(x)$ is such that $|f''(x)| \le 9$ for $0 \le x \le 10$. Find the smallest value of n such that when using the trapezoid approximation to compute $\int_1^6 f(x)dx$ the actual error E_T satisfies $|E_T| \le 5(10^{-3})$.

$$\frac{9(6-1)^3}{12n^2} \le 5(10^{-3})$$

$$\frac{4n^2}{3(125)} \ge \frac{5}{10^3}$$

$$n^2 \ge 18750$$

$$n \ge 136.9$$

n must be an integer

$$n = 137$$

6555 Review Improper Integrals

1. Explain why the following integral is improper. If the following integral is convergent, then find its value. If it is divergent, explain why it is divergent.

FOR $t > 2$

$$\int_2^t \frac{x\,dx}{(x^2+5)^{3/2}} = \frac{1}{3} - \frac{1}{\sqrt{t^2+5}}$$

$$\int_2^\infty \frac{x\,dx}{(x^2+5)^{3/2}}$$

$$\lim_{t \to +\infty} \left[\frac{1}{3} - \frac{1}{\sqrt{t^2+5}} \right] = \frac{1}{3}$$

the integral is improper because it has ∞ as a limit of integration.

$$\int_2^\infty \frac{x\,dx}{(x^2+5)^{3/2}} = \frac{1}{3}$$

the limit exists and so the integral is convergent.

2. Explain why the following integral is an improper integral. Evaluate it.

FOR $4 < t < \infty$

$$\int_t^{10} \frac{dx}{(x-4)^{3/2}} = \frac{2}{\sqrt{t-4}} - \frac{2}{\sqrt{6}}$$

$$\int_4^{10} \frac{dx}{(x-4)^{3/2}}$$

$$\lim_{t \to 4+} \frac{2}{\sqrt{t-4}} - \frac{2}{\sqrt{6}} = +\infty$$

the integral is improper because the integrand $(x-4)^{-3/2}$ is not defined at the lower limit of integration namely 4.

the limit does not exist, the integral is divergent.

3. Evaluate the following indefinite integral showing all steps.

$$\int \sin^2(5x)\cos^2(5x)\,dx$$

$$\frac{1}{4}\int [1 - \cos(10x)][1 + \cos(10x)]\,dx$$

$$= \frac{1}{8}\int [1 - \cos(20x)]\,dx = \frac{x}{8} - \frac{1}{160}\sin(20x) + c$$

4. Set up the integral which when evaluated will give the surface area of the surface obtain when the curve $y = (3x+7)^{1/2}$ from point where $x = 1$ to the point where $x = 7$ is revolved about the x axis.

$$2\pi \int_1^7 \sqrt{3x+7}\left(\frac{1}{2}\right)\sqrt{\frac{12x+37}{3x+7}}\,dx = \pi \int_1^7 \sqrt{12x+37}\,dx$$

6556 Review: Differential Equations

1. Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{2y+3}{x+5}$ and $y(0) = 1$.

$$\frac{2\,dy}{2y+3} = \frac{2\,dx}{x+5}$$

$$\ln|2y+3| = 2\ln|x+5| + \ln c \qquad c > 0$$

$$2y+3 = c(x+5)^2 \qquad c \neq 0$$

$$y = -\frac{3}{2} + \frac{1}{10}(x+5)^2$$

2. Solve the initial value problem $\frac{dy}{dt} = (2-y)(10-y)$ and $y(0) = 4$.

$$\left(\frac{1}{2-y} - \frac{1}{10-y}\right) dy = 8\,dt$$

$$\ln\left|\frac{10-y}{2-y}\right| = \ln c\,e^{8t}$$

$$\frac{10-y}{2-y} = c\,e^{8t}$$

$$y = \frac{2c\,e^{8t} - 10}{c\,e^{8t} - 1}$$

$$y(0) = 4 \text{ gives } c = -3$$

$$y = \frac{10 + 6e^{8t}}{1 + 3e^{8t}} = \frac{6 + 10e^{-8t}}{3 + e^{-8t}}$$

3. A glass plate in the side of a water tank has the shape shown below. The top of the plate is 4 feet below the surface of the water. Find the hydrostatic force exerted by the water on this glass plate. The length units are in feet and water density is 62.5 lb/ft³. Hint: Begin by finding the equation of the sloping line.
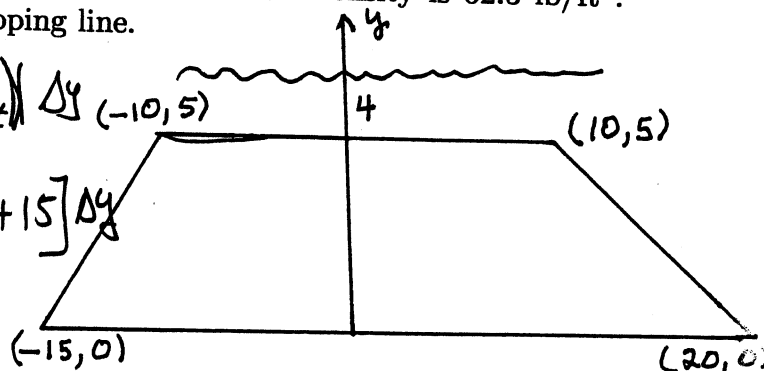

$$\Delta F = \rho(\text{depth})(X_{Right} - X_{left})\,\Delta y$$

$$\Delta F = \rho(9-y)[20 - 2y - y + 15]\,\Delta y$$

$$\text{Force} = \rho \int_0^5 (9-y)(35 - 3y)\,dy$$

$$= 925\rho$$

$$= 57{,}812.5 \text{ lbs}$$

(-10,5)   4   (10,5)

(-15,0)   (20,0)

$$y = x + 15$$
$$x = y - 15$$

$$y = -\frac{1}{2}x + 10$$
$$x = 20 - 2y$$