6533 Improper Integrals

1. Given that $\int \frac{-2x^4+18x}{(2x^3+9)^2}dx = \frac{x^2}{2x^3+9} + C$, find $\int_1^t \frac{-2x^4+18x}{(2x^3+9)^2}dx$ and $\int_1^\infty \frac{-2x^4+18x}{(2x^3+9)^2}dx$.

$$\int_1^t \frac{-2x^4+18x}{(2x^3+9)^2}dx = \frac{t^2}{3t^3+9} - \frac{1}{11}$$

$$\int_1^\infty \frac{-2x^4+18x}{(2x^3+9)^2}dx = \lim_{t\to+\infty}\left(\frac{t^2}{2t^3+9} - \frac{1}{11}\right) = -\frac{1}{11}$$

2. Given that $\int \frac{dx}{\sqrt{4-x}} = -2\sqrt{4-x}+C$, for $x<4$, find $\int_0^t \frac{dx}{\sqrt{4-x}}$ when $0<t<4$, and then $\int_0^4 \frac{dx}{\sqrt{4-x}}$.

$$\int_0^t \frac{dx}{\sqrt{4-x}} = 4 - 2\sqrt{4-x} \quad \text{for } 0<t<4$$

$$\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{t\to4^-}\left[4-2\sqrt{4-t}\right] = 4$$

3. Explain why the following integral is improper. If it converges, then evaluate it. If it diverges, explain why.

$$\int_0^4 \frac{dx}{(4-x)^{3/2}}$$

the integral is improper because the integrand is not defined at one end point of the interval of integration.

$$\int_0^t \frac{dx}{(4-x)^{3/2}} = 2(4-t)^{-1/2} - 1$$

for $0<t<4$

$$\int_0^4 \frac{dx}{(4-x)^{3/2}} = \lim_{t\to4^-}\left[2(4-t)^{-1/2}-1\right] = +\infty$$

the limit does not exist. the integral is divergent,

4. If the following improper integral converges then evaluate it. If it diverges, then explain why it diverges.

$$\int_1^\infty \frac{x\,dx}{(4+x^2)^{3/2}}$$

the integral is improper because the upper limit is ∞ rather than an actual real number,

$$\int_1^t \frac{x\,dx}{(4+x^2)^{3/2}} = \frac{1}{\sqrt5} - \frac{1}{\sqrt{4+t^2}}$$

$$\int_1^\infty \frac{x\,dx}{(4+x^2)^{3/2}} = \lim_{t\to+\infty}\left(\frac{1}{\sqrt5} - \frac{1}{\sqrt{4+t^2}}\right) = \frac{1}{\sqrt5}$$

the integral converges and its value is $\frac{1}{\sqrt5}$

6534 Error Bounds

1. If $|f''(x)| \leq 3$ for $1 \leq x \leq 5$, find the smallest value of n such that E_T the error when approximating $\int_1^5 f(x)dx$ using the trapezoid rule satisfies $|E_T| \leq 10^{-4}$.

$$\frac{K(b-a)^3}{12n^2} = \frac{3(5-1)^3}{12n^2} = \frac{16}{n^2}$$

$$\frac{16}{n^2} \leq 10^{-4}$$

$$\frac{n^2}{16} \geq 10^4$$

$$n \geq 4(10^2)$$

the smallest value of n is $n = 400$

2. Suppose we know that $|f''(x)| \leq (1/2)$. Consider the integral $\int_1^4 f(x)dx$. Find the smallest value of the integer n such that

$$\frac{K(b-a)^3}{12n^2} < 10^{-3}$$

$$\frac{\frac{1}{2}(4-1)^3}{12n^2} \leq 10^{-3}$$

$$\frac{8n^2}{9} \geq 10^3$$

$$n^2 \geq \frac{9(10^3)}{8} = 1125$$

$$n \geq 33.5$$

n is an integer

the smallest value of n is

$n = 34$

3. Let $f(x) = (1/12)x^4 - x^3 + 8x + 4$. Find a bound for the error E_T for the trapezoid rule for the integral $\int_0^5 f(x)dx$ when $n = 125$.

$f''(x) = x^2 - 6x$

$f'''(x) = 2x - 6$

$f'''(3) = 0$

$f''''(3) = 2$

Local min at $x = 3$

$f''(3) = -9$

$f'''(0) = -6$ $f''(x)$ has local max at $= 0$

$f'''(5) = 4$ $f''(x)$ has local max at $x = 5$

$f''(0) = 0$ $f''(5) = -5$

$|f''(x)| \leq 9$ for $0 \leq x \leq 5$

$$|E_T| \leq \frac{(9)(5-0)^3}{12(125)^2} = \frac{3}{500} = 0.006$$

4. If $b = 12$, $a = 2$, and $K = 24$, find the smallest value of n such that $\frac{K(b-a)^3}{12n^2} \leq 10^{-3}$. Recall n is an integer.

$$\frac{K(b-a)^3}{12n^2} = \frac{24(10)^3}{12n^2} = \frac{2(10)^3}{n^2}$$

$$\frac{2(10)^3}{n^2} \leq 10^{-3}$$

$$n^2 \geq 2(10^6)$$

$$n \geq 1414.2$$

the smallest value of n is

$n = 1415$

6535 Harder Integrals

1. Find the following antiderivative: Hint: start with "let $u^2 =$". Assume $x \geq 0$.

$$\int \frac{dx}{x\sqrt{2x+9}}$$

Let $u^2 = 2x+9$ $\quad x = \frac{u^2-9}{2}$ $\quad dx = u\,du$

$$2\int \frac{du}{u^2-9} = \frac{1}{3}\int \left[\frac{1}{u-3} - \frac{1}{u+3}\right] du$$

$$= \frac{1}{3}\left[\ln|u-3| - \ln|u+3|\right] + C$$

$$= \frac{1}{3}\left[\ln\left|\sqrt{2x+9} - 3\right| - \ln\left|\sqrt{2x+9} + 3\right|\right] + C$$

If $x > 0$ then $(\sqrt{2x+9} > 3$

2. Find the antiderivative $\int \tan^3(5x)\sec(5x)dx$

$$\int \left(\sec^2(5x) - 1\right)\left[\tan(5x)\sec(5x)\right] dx$$

$$= \frac{\sec^3(5x)}{15} - \frac{\sec(5x)}{5} + C$$

$u = \sec(5x)$ gives $\frac{1}{5}\int (u^2-1)\,du$

3. Suppose $f(x)$ is such that $|f''(x)| \leq 6$ for $1 \leq x \leq 6$. Find the bound for the error E_T for the integral $\int_1^6 f(x)$ using $n = 50$. Recall $|E_T| \leq \frac{K(b-a)^3}{12n^2}$.

$$\frac{(K)(b-a)^3}{12n^2} = \frac{6(6-1)^3}{12(50)^2} = \frac{1}{40}$$

the Actual error E_T satisfies

$$|E_T| \leq \frac{1}{40} = 0.025$$

6536 Integrals Using Trigonometric Identities

1. Recall the identity $\tan^2 x + 1 = \sec^2 x$. Find the antiderivative

$$\int \tan^4 x \, dx.$$

$$\int (\tan^2 x \sec^2 x - \sec^2 x + 1)\, dx$$

$$= \frac{\tan^3 x}{3} - \tan x + x + C$$

2. Find the antiderivative

$$\int \tan x \sec^4 x \, dx$$

$$\int (\tan x + \tan^3 x) \sec^2 x \, dx$$

$$\frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C$$

3. Explain why each of the following integrals is improper.

a) $\displaystyle\int_0^\infty \frac{x\,dx}{\sqrt{1+x^2}}$ b) $\displaystyle\int_2^5 \frac{x+2}{\sqrt{5-x}}dx$

d) the integral has $-\infty$ as a limit.

c) $\displaystyle\int_2^5 \frac{x+2}{(x-2)^{3/2}}dx$ d) $\displaystyle\int_{-\infty}^4 xe^{-x}dx$

a) the integral has ∞ as a limit
b) the integrand $\frac{x+2}{\sqrt{5-x}}$ is not defined at the upper limit $x=5$.
c) the integrand $\frac{x+2}{(x-2)^{3/2}}$ is not defined at the lower limit $x=2$,