Solution to worksheet 9913.4

① $$y_{n+1} = 1.03 y_n + 100 \ , \quad y_0 = 100.$$

To find y_n, one can use the formula on page 522 of the textbook. Another method is to mimick the way we solve a linear differential equation: First, find homogeneous solution to $y_{n+1} - 1.03 y_n = 0$.

Characteristic equation; $\lambda - 1.03 = 0$, which has root $\lambda = 1.03$. Thus, $y_n^h = C(1.03)^n$.

Secondly, find a particular solution: $y_n^p = A$ (constant, independent of n).

$$A = 1.03 A + 100$$

$$\Rightarrow A = -\frac{100}{0.03}$$

Then $$y_n = y_n^h + y_n^p = C(1.03)^n - \frac{100}{0.03}.$$

Then condition $y_0 = 100$ helps us determine C.

Answer: $y_{20} = \$2867.65$

(2) Characteristic equation $\lambda^2 - 5\lambda + 6 = 0$, which has two solutions $\lambda = 2$ and $\lambda = 3$. Thus, the general solution is $y_n = C_1 2^n + C_2 3^n$. The conditions $y_1 = 1$, $y_2 = 4$ determine $C_1 = -\frac{1}{2}$ and $C_2 = \frac{2}{3}$.

(3) $y_n - y_{n-1} = n^2$.

Homogeneous equation $y_n - y_{n-1} = 0$ has characteristic equation $\lambda - 1 = 0$, which has root $\lambda = 1$. Thus, homogeneous solution is $y_n^h = C \cdot 1^n = C$.

Based on the right hand side, we guess a particular solution to be of the form

$$y_n^p = An^2 + Bn + D.$$

However, the part D appears in y_n^h. We adjust the first guess by multiplying y_n^p by n:

$$y_n^p = n(An^2 + Bn + D) = An^3 + Bn^2 + Dn.$$

Substituting this y_n^p into the equation $y_n - y_{n-1} = n^2$, we get $A = \frac{1}{3}$, $B = \frac{1}{2}$, $D = \frac{1}{6}$. Thus,

$$y_n = y_n^h + y_n^p = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + C$$

The condition $y_0 = 0$ gives $C = 0$.