Solutions to Worksheet 7841 (Least squares)

1) Suppose we want to find a line $y = mx + b$ that fits best the points $(1,6), (2,9), (4,11), (5,12)$ and $(6,13)$.

Ideally, m and b satisfy

$$6 = m.1 + b = [1 \quad 1]\begin{bmatrix} m \\ b \end{bmatrix}$$

$$9 = m.2 + b = [2 \quad 1]\begin{bmatrix} m \\ b \end{bmatrix}$$

$$\cdots$$

Put these equations together:

$$\begin{bmatrix} 6 \\ 9 \\ 11 \\ 12 \\ 13 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \\ 6 & 1 \end{bmatrix}}_{B} \begin{bmatrix} m \\ b \end{bmatrix}$$

Multiply the transpose of B to the left of each side:

$$\underbrace{\begin{bmatrix} 1 & 2 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 9 \\ 11 \\ 12 \\ 13 \end{bmatrix}}_{\begin{bmatrix} 206 \\ 51 \end{bmatrix} \cancel{\begin{bmatrix} 82 & 18 \\ 18 & 5 \end{bmatrix}}} = \underbrace{\begin{bmatrix} 1 & 2 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \\ 6 & 1 \end{bmatrix}}_{\begin{bmatrix} 82 & 18 \\ 18 & 5 \end{bmatrix}} \begin{bmatrix} m \\ b \end{bmatrix}$$

Thus,

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 82 & 18 \\ 18 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 206 \\ 51 \end{bmatrix} = \begin{bmatrix} 1.3023 \\ 5.5116 \end{bmatrix}.$$

2) $$y' + \underbrace{\frac{1}{3t+4}}_{p(t)} y = 7$$

$$P(t) = \int \frac{1}{3t+4}\,dt = \frac{1}{3}\ln(3t+4) = \ln(3t+4)^{\frac{1}{3}}$$

$$e^{P(t)} = (3t+4)^{\frac{1}{3}}.$$

Multiply both sides of the differential equation by $(3t+4)^{\frac{1}{3}}$:

$$\left[y(3t+4)^{\frac{1}{3}}\right]' = 7(3t+4)^{\frac{1}{3}}$$

Integrate both sides:

$$y(3t+4)^{\frac{1}{3}} = \int 7(3t+4)^{\frac{1}{3}}dt = \frac{\cancel{21}7}{4}(3t+4)^{\frac{4}{3}} + C$$

Then

$$y = \frac{\cancel{21}7}{4}(3t+4) + \frac{C}{(3t+4)^{1/3}}$$

Because $y(0) = 10$, $C = 3 \cdot 4^{\frac{1}{3}}$.

Solutions to Worksheet 7842 (Utopium)

1) Let k be the decay constant. Then

$$\frac{dy}{dt} = -ky + \underset{\uparrow}{1}$$

the rate contributed by the nuclear reactor

To find k, we write the equation of decaying without artificial intervention:

$$\frac{dQ}{dt} = -kQ$$

Then $Q(t) = Q_0 e^{-kt}$. Since utopium takes 7 years to decay to 50% of its original amount,

$$Q(7) = \frac{1}{2} Q_0 \Rightarrow e^{-k \cdot 7} = \frac{1}{2} \Rightarrow \boxed{k = \frac{\ln 2}{7}}$$

Return to the equation

$$y' + ky = 1$$

Integration factor is e^{kt}. We get $\left(y(t) e^{kt}\right)' = e^{kt}$.

Integrating both sides:

$$y e^{kt} = \int e^{kt} dt = \frac{1}{k} e^{kt} + C.$$

Thus, $y = \frac{1}{k} + C e^{-kt}$.

Since $y(0) = 2$, $C = 2 - \frac{1}{k} = 2 - \frac{7}{\ln 2}$.

2) a.

$$\begin{bmatrix} 4 & 2 \\ 6 & 2 \\ 12 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 5 \\ 3 \end{bmatrix}$$

b. Think of x and y as m and k when we try to fit the points (what are they? There are 4 of them).

$$\begin{bmatrix} 4 & 6 & 12 & 5 \\ 2 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 6 & 2 \\ 12 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & 6 & 12 & 5 \\ 2 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 5 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 221 & 61 \\ 61 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 77 \\ 18 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 221 & 61 \\ 61 & 18 \end{bmatrix}^{-1} \begin{bmatrix} 77 \\ 18 \end{bmatrix} = \begin{bmatrix} 1.1206 \\ -2.7977 \end{bmatrix}$$