9974 - Eigenstuff

1) $\det(A-\lambda I_2) = \lambda^2 + 2\lambda + 5$ has two complex roots $\lambda = -1 \pm 2i$

- Find eigenvectors of $\lambda_1 = -1-2i$:

$$A - \lambda_1 I_2 = \begin{bmatrix} -10+2i & 4 \\ -26 & 10+2i \end{bmatrix} \xrightarrow[R_2 \to R_2 + 26R_1]{R_1 \to \frac{R_1}{-10+2i}} \begin{bmatrix} 1 & \frac{5-i}{13} \\ 0 & 0 \end{bmatrix}$$

$$c_2 = t, \quad c_1 = -\frac{5-i}{13} c_2 = -\frac{5-i}{13} t, \quad v_1 = \begin{bmatrix} t \\ -\frac{5-i}{13} t \end{bmatrix}$$

- Find eigenvectors of $\lambda_2 = -1+2i$:

$$A - \lambda_2 I_2 = \begin{bmatrix} -10-2i & 4 \\ -26 & 10-2i \end{bmatrix} \xrightarrow[R_2 \to R_2 + 26R_1]{R_1 \to \frac{R_1}{-10-2i}} \begin{bmatrix} 1 & \frac{-5+i}{13} \\ 0 & 0 \end{bmatrix}$$

$$c_2 = t, \quad c_1 = -\frac{-5+i}{13} t, \quad v_2 = \begin{bmatrix} t \\ -\frac{-5+i}{13} t \end{bmatrix}$$

2) $\det(A-\lambda I_2) = \lambda^2 - 2\lambda + 1$ has double root $\lambda_0 = 1$.

Find corresponding eigenvectors:

$$A - \lambda_0 I_2 = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix}$$

$$c_2 = t, \quad c_1 = -\frac{3}{2} c_2 = -\frac{3}{2} t, \quad v = \begin{bmatrix} t \\ -\frac{3}{2} t \end{bmatrix}$$

Eigenspace is the set $\left\{ \begin{bmatrix} t \\ -\frac{3}{2} t \end{bmatrix} : t \in \mathbb{R} \right\}$.

3) $\det(A-\lambda I_3) = -(\lambda^3 - 5\lambda^2 + 8\lambda - 4) = -(\lambda-1)(\lambda-2)^2$

has roots $\lambda_1 = 1$ and $\lambda_2 = 2$.

- Find eigenspace corresponding to λ_1:

$$A - \lambda_1 I_3 = \begin{bmatrix} -2 & -3 & 3 \\ 0 & 1 & 0 \\ -2 & -2 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$c_3 = t$, $c_2 = 0$, $c_1 = \frac{3}{2}c_3 = \frac{3}{2}t$.

$$\text{eigenspace} = \left\{ \begin{bmatrix} 3/2\, t \\ 0 \\ t \end{bmatrix} : t \in \mathbb{R} \right\}.$$

- Find eigenspace corresponding to λ_2:

$$A - \lambda_2 I_3 = \begin{bmatrix} -3 & -3 & 3 \\ 0 & 0 & 0 \\ -2 & -2 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$c_3 = t$, $c_2 = s$, $c_1 = c_3 - c_2 = t - s$.

$$\text{eigenspace} = \left\{ \begin{bmatrix} t-s \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

9973 - More undetermined coefficients

1) $$y'' + 16y = -\sin(4t) + 3e^{5t}.$$

First, general solution to the homogeneous equation is

$$y_h = C_1\cos(4t) + C_2\sin(4t)$$

We observe that the term $\sin(4t)$ in y_h appears on the right hand side, so y_p should contain

$$t\big(A\cos(4t) + B\sin(4t)\big).$$

Since e^{5t} also appears on the right hand side, y_p should also contain Ce^{5t}. Thus, our guess is

$$y_p = t(A\cos 4t + B\sin 4t) + Ce^{5t}.$$

Now substitute y_p into the original equation to determine A, B, C:

optional

$$y'' + 16y = 8B\cos(4t) - 8A\sin(4t) + 41Ce^{5t}$$

$$\begin{cases} 8B = 0 \\ 8A = 1 \\ 41C = 3 \end{cases} \Longrightarrow \begin{cases} B = 0 \\ A = 1/8 \\ C = 3/41 \end{cases}$$

Thus, $$y_p = \frac{t}{8}\cos(4t) + \frac{3}{41}e^{5t}$$

2) $y'' - 2y' + y = te^t + \cos(2t)$

General solution to homogeneous equation is $y_h = C_1 e^t + C_2 te^t$.

First, we see the term te^t on the right hand side, so we guess y_p should contain $(At+B)e^t$. However, the part Be^t appears in y_h. So we ~~multi~~ adjust our guess: saying y_p should contain $t(At+B)e^t$. However, the term Bte^t also appears in y_h. So we keep adjusting our guess by multiplying by t again: say y_p should contain $t^2(At+B)e^t$. This guess is good because no terms appear in y_h.

Since the right hand side has $\cos(2t)$, y_p should also contain $C\cos(2t) + D\sin(2t)$. Therefore, our guess is

$$y_p = t^2(At+B)e^t + C\cos(2t) + D\sin(2t)$$

optional

Substitute into the differential equation to find A, B, C, D:

$$y'' - 2y' + y = (6At + 2B)e^t + (-3C - 4D)\cos 2t + (4C - 3D)\sin 2t$$

$$\begin{cases} 6A = 1 \\ 2B = 0 \\ -3C - 4D = 1 \\ 4C - 3D = 0 \end{cases} \Longrightarrow \begin{cases} A = 1/6 \\ B = 0 \\ C = -3/25 \\ D = -4/25 \end{cases}$$

3) $y'' + 0.2y' = -32$

Characteristic equation $r^2 + 0.2r = 0$ has two roots $r_1 = 0$ and $r_2 = -0.2$. General solution of the homogeneous equation is $y_h = C_1 e^{0t} + C_2 e^{-0.2t} = C_1 + C_2 e^{-0.2t}$.

Now we guess a particular solution: The right hand side is just a constant, so we might guess $y_p = A$. However, this guess doesn't work become the equation becomes $0 = 32$! Thus, we guess $y_p = At$. The equation then becomes $0.2A = -32$, which gives $A=-160$.

Thus,

$$y = y_h + y_p = C_1 + C_2 e^{-0.2t} - 160t.$$

Invoke the ~~int~~ initial ~~data~~ conditions $y(0)=100$ and $y'(0)=0$. We get $C_1 = 900$ and $C_2 = -800$.