9993 – 2x2 systems with complex eigenvalues

(1) $$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{X'} = \underbrace{\begin{bmatrix} -18 & 35 \\ -14 & 24 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X}$$

Eigenvalues of A are $\lambda_1 = 3+7i$, and $\lambda_2 = 3-7i$.

An eigenvector corresponding to λ_1 is $\vec{v}_1 = \begin{bmatrix} \frac{3}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}.$$

We have

$$e^{\lambda_1 t} \vec{v}_1 = e^{(3+7i)t} \left(\begin{bmatrix} 3/2 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \right)$$

$$= e^{3t} (\cos 7t + i \sin 7t) \left(\begin{bmatrix} 3/2 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \right)$$

$$= \underbrace{e^{3t} \left(\cos 7t \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} - \sin 7t \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \right)}_{X_1}$$

$$+ i \underbrace{e^{3t} \left(\sin 7t \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} + \cos 7t \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \right)}_{X_2}$$

$$= X_1 + i X_2.$$

General solution is $X = C_1 X_1 + C_2 X_2$

$$= C_1 e^{3t} \left(\cos 7t \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} - \sin 7t \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \right) + C_2 e^{3t} \left(\sin 7t \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} + \cos 7t \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \right).$$

2 $\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{X'} = \underbrace{\begin{bmatrix} -13 & 10 \\ -20 & 15 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X}$, $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$

Eigenvalues of A are $\lambda_1 = 1+2i$ and $\lambda_2 = 1-2i$.

An eigenvector corresponding to λ_1 is

$$\vec{v}_1 = \begin{bmatrix} 7-i \\ 10 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

We have

$$e^{\lambda_1 t}\vec{v}_1 = e^{(1+2i)t}\left(\begin{bmatrix} 7 \\ 10 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

$$= e^{t}(\cos 2t + i \sin 2t)\left(\begin{bmatrix} 7 \\ 10 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

$$= \underbrace{e^{t}\left(\cos 2t \begin{bmatrix} 7 \\ 10 \end{bmatrix} - \sin 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)}_{X_1} + i \underbrace{e^{t}\left(\cos 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin 2t \begin{bmatrix} 7 \\ 10 \end{bmatrix}\right)}_{X_2}$$

General solution is

$$X = C_1 X_1 + C_2 X_2$$

$$= C_1 e^{t}\left(\cos 2t \begin{bmatrix} 7 \\ 10 \end{bmatrix} - \sin 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) + C_2 e^{t}\left(\cos 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin 2t \begin{bmatrix} 7 \\ 10 \end{bmatrix}\right)$$

Plug $t=0$ and use the initial condition $X(0) = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$ to

obtain $C_1 = -\frac{1}{2}$ and $C_2 = -\frac{27}{2}$.

9994 - Reconciling two methods of solution

(1)
$$\underbrace{\begin{bmatrix} u' \\ w' \end{bmatrix}}_{X'} = \underbrace{\begin{bmatrix} 0 & 1 \\ -58 & -6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u \\ w \end{bmatrix}}_{X}, \quad \begin{bmatrix} u(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Eigenvalues $\lambda_1 = -3+7i$, $\lambda_2 = -3-7i$.

Eigenvector corresponding to λ_1:
$$\vec{v}_1 = \begin{bmatrix} -3-7i \\ 58 \end{bmatrix} = \begin{bmatrix} -3 \\ 58 \end{bmatrix} + i\begin{bmatrix} -7 \\ 0 \end{bmatrix}.$$

we have
$$e^{\lambda_1 t}\vec{v}_1 = e^{(-3+7i)t}\left(\begin{bmatrix} -3 \\ 58 \end{bmatrix} + i\begin{bmatrix} -7 \\ 0 \end{bmatrix}\right)$$

$$= X_1 + iX_2$$

where
$$X_1 = e^{-3t}\left(\cos 7t \begin{bmatrix} -3 \\ 58 \end{bmatrix} - \sin 7t \begin{bmatrix} -7 \\ 0 \end{bmatrix}\right)$$

$$X_2 = e^{-3t}\left(\cos 7t \begin{bmatrix} -7 \\ 0 \end{bmatrix} + \sin 7t \begin{bmatrix} -3 \\ 58 \end{bmatrix}\right)$$

General solution: $X = C_1 X_1 + C_2 X_2$.

Use the condition $X(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ to determine
$$C_1 = -3/58, \quad C_2 = -107/406.$$

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(2) $y = e^{-3t}\left(2\cos 7t + \frac{3}{7}\sin 7t\right)$

(3) $\begin{cases} u = y \\ w = y' \end{cases} \Rightarrow \begin{cases} u' = y' = w \\ w' = y'' = -6y' - 58y = -6w - 58u \end{cases}$

$\Rightarrow \begin{bmatrix} u' \\ w' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -58 & -6 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}.$