HOMEWORK #1 (DUE FRIDAY, SEPT. 23).

9/11/2011

Note: *Turn in only the "starred" problems; out of these, only selected problems will be graded.*

1.* Find all subgroups of the additive group $\mathbb{Z} \times \mathbb{Z}$.

2. Let G be group and let H be a subgroup in G.

(a) Show that there is a bijection between the sets of left and right cosets of H in G. In particular, one can define the index $(G : H)$ as the cardinality of the set of right cosets.

(b) Show that a subgroup of index two is always normal.

3.* Let G be a group, H a subgroup in G, and let N_H be the normalizer of H.

(a) Show that if $K < G$ is a subgroup such that H is a normal subgroup of K, then $K \subset N_H$, i.e., N_H is the largest subgroup of G in which H is normal.

(b) If K is a subgroup contained in N_H, then KH is a group and H is a normal subgroup in KH.

(c) If G is finite and $K \subset N_H$, then

$$|KH| = \frac{|H||K|}{|H \cap K|}.$$

4. Determine all (nonisomorphic) finite groups with order at most 8.

5.* Problem 7, page 75 in Lang.

6.* Problem 9, page 75 in Lang.

7.* (Divisible groups) An abelian group $(G, +)$ is said to be *divisible* if for any $y \in G$ and $n \in \mathbb{Z}$, $n \neq 0$, there is an x in G with $nx = y$. (The simplest example is $(\mathbb{Q}, +)$.)

(a) Show that any divisible group G is infinite, and that G has no subgroups of finite index other than G itself.

(b) Let $U = \mathbb{Q}/\mathbb{Z}$. Show that every element of U is a torsion element, that is, every element has finite order (or finite period, in the terminology in Lang's book). For each $n \geq 1$ show that U has a unique subgroup of order n, and that this subgroup is cyclic.

(c) For a prime p, let U_p be the sugroup of U consisting of all p-torsion elements, that is, all elements whose order is a power of p. Show that U_p is a divisible group, and describe all its subgroups.

(**Remark:** We'll revisit divisible groups (in a more general context) towards the end of the Spring semester when we'll do a bit of Homological Algebra, and we'll use the results in this problem at that time. We'll also show then that any divisible group is a direct sum of copies of $\mathbb{Q}$ and U_p for various p.)

8.* Let G be a finite abelian group which is *not* cyclic. Prove that there is a prime number p and a subgroup H of G with $H \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.