HOMEWORK #3 (DUE WEDNESDAY, OCT. 19; TURN IN *ALL* PROBLEMS).

10/11/2011

1. Let A be a commutative ring.

(a) Show that a formal power series $f = \sum_{i=0}^{\infty} a_i X^i \in A[[X]]$ is invertible if and only if a_0 is invertible in A.

(b) Show that a polynomial $f = \sum_{i=0}^{n} a_i X^i \in A[X]$ is invertible if and only if a_0 is invertible in A and $a_1, \ldots, a_n$ are all nilpotent. (An element a in a commutative ring A is nilpotent if $a^m = 0$ for some $m \geq 1$.)

2. Let $(G, +)$ be a finite abelian group whose order is not divisible by the square of any integer. (We say the order is *square-free.*)

(a) Show that, up to isomorphism, there is a unique structure of ring with 1 on G. (That is, show that one can define multiplications on the set G, which together with the given additive operation form structures of ring with 1, then show that all these rings are isomorphic.)

(b) Let A denote the ring $(G, +, \cdot)$ from part (a) above. Prove that the multiplicative group $A[X]^*$ of invertible polynomials in $A[X]$ coincides with the multiplicative group A^* of invertible elements in A. (Problem $1(b)$ is relevant here; Problem $5(i), (ii)$ might help, but you don't really need it.)

(c) Show by examples that both (a) and (b) are false if we don't assume that the order of G is not divisible by a square.

3. Problem 8, page 115 in Lang.

4. Let d be a square-free nonzero integer.

(i) Show that

$$\mathbb{Z}[\sqrt{d}] := \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$$

is an integral domain, isomorphic to $\mathbb{Z}[X]/(X^2 - d)$.

(ii) If $x = a + b\sqrt{d}$, find a generator t of the ideal $(x) \cap \mathbb{Z}$, and a generator r of the ideal $\{m \in \mathbb{Z} \mid m\sqrt{d} \in (x)\}$.

(iii) Is it true in general that $(x) = t\mathbb{Z} + r\sqrt{d}\mathbb{Z}$?

5. Let A, B be commutative rings and let $\mathfrak{N}(A)$ (respectively $\mathfrak{N}(B)$) denote the set of nilpotent elements in A (respectively B).

(i) Show that $\mathfrak{N}(A)$ is an ideal.

(ii) Show that $\mathfrak{N}(A \times B) = \mathfrak{N}(A) \times \mathfrak{N}(B)$.

(iii) Show that $\mathfrak{N}(A)$ is the intersection of all prime ideals in A. (One inclusion is easy, for the other you'll need to use Zorn's Lemma. Come ask me if you have trouble with this.)