HOMEWORK #4 (DUE FRIDAY, OCT. 28).

10/21/2011

Turn in the starred problems only.

1. Let A be a commutative ring and let $S \subset A$ be a multiplicative system which does not contain zero divisors. Show that the canonical inclusion $S^{-1}(A[X_1, \dots, X_n]) \subset (S^{-1}A)[X_1, \dots, X_n]$ is an isomorphism, but in general the inclusion $S^{-1}(A[[X]]) \subset (S^{-1}A)[[X]]$ is not bijective.

2. Let A be a commutative ring and let $f \in A$ be an element which is not a zero divisor. Show that $A_f \cong A[X]/(fX - 1)$. (Recall that A_f denotes the ring of fractions $S^{-1}A$ for $S = \{1, f, f^2, \dots\}$.)

3. Let $\phi : A \subset B$ be a morphism of commutative rings, S a multiplicative system in A, and T a multiplicative system in B with $\phi(S) \subset B$. Show that there exists an unique morphism $\phi' : S^{-1}A \to T^{-1}B$ such that $\phi i_T = i_S \phi'$, where $i_S : A \to S^{-1}A$ and $i_T : B \to T^{-1}B$ are the canonical morphisms.

4*. Let A be a commutative ring and let $S \subset A$ be a multiplicative system which does not contain zero divisors. Put

$$T = \{x \in A \mid \text{there are } s \in S, y \in A \text{ with } s = xy\}.$$

(i) Show that T is a multiplicative system with the property that $xy \in T$ iff $x \in T$ and $y \in T$. (In general a multiplicative system with this property is called *saturated*. The system T defined above is called the *saturation of* S.)

(ii) Show that $S^{-1}A \cong T^{-1}A$.

5. Let A be a commutative ring and let $S \subset A$ be a multiplicative system which does not contain zero divisors. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in A. The subset

$$S^{-1}\mathfrak{a} = \left\{ \frac{a}{s} \mid a \in \mathfrak{a}, s \in S \right\}$$

is obviously an ideal in $S^{-1}A$. Prove the following:

(i) If $\mathfrak{a} \subset \mathfrak{b}$ then $S^{-1}\mathfrak{a} \subset S^{-1}\mathfrak{b}$.

(ii) $S^{-1}(\mathfrak{a} + \mathfrak{b}) = S^{-1}\mathfrak{a} + S^{-1}\mathfrak{b}$.

(iii) $S^{-1}(\mathfrak{a} \cap \mathfrak{b}) = S^{-1} \cap S^{-1}\mathfrak{b}$.

6*. Let A be an integral domain and let S be a multiplicative system with $0 \notin S$.

(a) If A is euclidean, is the ring of fractions $S^{-1}A$ euclidean as well? If $S^{-1}A$ is euclidean, must A be euclidean?

(b) Same questions as in (a) if we replace "euclidean" by "principal ideal domain".

(c) Same questions as in (a) if we replace "euclidean" by "unique factorization domain".

7*. Let A be a factorial ring which is not a field and assume that the multiplicative group of invertible elements in A is finite. Show that there are infinitely many (non-associate) irreducible elements in A.

8*. Describe all the prime ideals in $\mathbb{Z}[X]$.
9*. Problem 1, page 114 in Lang.
10. Problem 2, page 115 in Lang.
11*. Problem 6, page 115 in Lang.