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Homework #2
Math 8201: General Algebra

(1) Let G be a subgroup of S_n. 10

(i) If $G \cap A_n = \{id\}$ then $|G| \le 2$

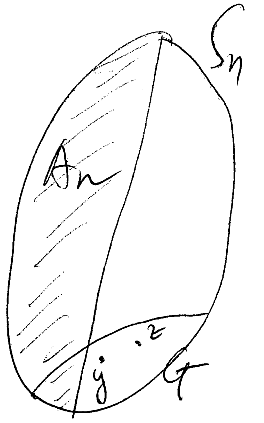
(ii) If $|G| > 2$ ~~then~~ and G is simple, then $G \subset A_n$.

(iii) If $n \geq 5$ then S_n has no subgroup of index m with $2 < m < n$.

(iv) If $n \geq 5$, then A_n has no subgroup of index m with $2 \le m < n$.

Proof

(i) Let G be a subgroup of S_n and $G \cap A_n = \{id\}$. Suppose that



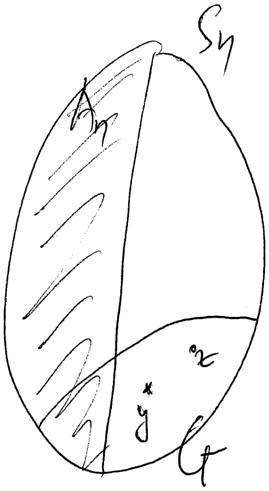
$|G| \geq 3$. Then there exist distinct elements y and z in $G \setminus A_n$. Since $y, z \notin A_n$, we have $yA_n, zA_n \neq A_n$. Since $|S_n/A_n| = 2$, i.e., $S_n/A_n = \{A_n, yA_n\}$, we have $zA_n = yA_n$.

Thus $z^{-1}y \in A_n$. It follows that $z^{-1}y \in A_n \cap G = \{e\}$.

Thus $z = y$, which is a contradiction.

(ii) Let G be a simple subgroup of S_n which has more than two ~~&~~ elements. Now we have ~~some~~ a single group, we should look for a normal

Subgroup to find something interesting. Our knowledge of A_n is that



A_n is normal in S_n and $|S_n/A_n| = 2$. Something might be normal is $A_n \cap G$. ~~we have $G \subset N_{A_n} = S_n$. Thus by the second isomorphism theorem,~~

Let $y, z \in G \setminus A_n$. Then since $S_n/A_n = \{A_n, yA_n\}$, we have $yA_n = zA_n$, or $z^{-1}y \in A_n$, or $z^{-1}y \in A_n \cap G$.

Thus $z(A_n \cap G) = y(A_n \cap G)$ and $G/(A_n \cap G) = \{A_n \cap G, y(A_n \cap G)\}$.

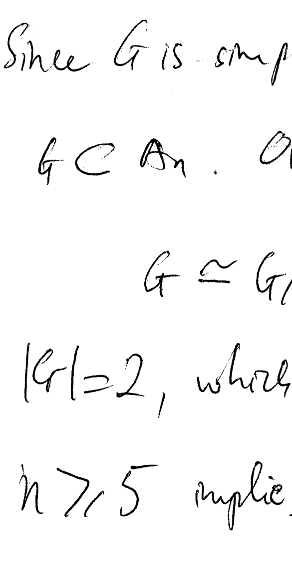
Thus $A_n \cap G$ has index 2 in G, and hence it is normal in G.

Since G is simple, $A_n \cap G$ is either $\{e\}$ or G. If $A_n \cap G = G$ then $G \subset A_n$. Otherwise, $A_n \cap G = \{e\}$ then thus

$$G \simeq G/\{e\} \simeq G/(A_n \cap G) \simeq S_2$$

and $|G| = 2$, which is a contradiction.

(iii) $n \geq 5$ implies A_n is simple group. Suppose S_n has a subgroup A of ~~order $2 < |A| < n$~~ index $2 < |S_n/A| < n$. Now that all what we



know about A is about the quotient S_n/A, which is only a set. Thus we need a group action to understand more about this set. An interesting group to act on S_n/A is A_n since it is simple.

Luckily, there is a natural action of A_n on S_n/A; that is the translation. We define $\pi : A_n \longrightarrow \mathrm{Perm}(S_n/A)$ be an action such that $x \in A_n \mapsto x : yA \mapsto xyA$ $\forall y \in S_n, x \in A_n$. Then π is a homomorphism. Since A_n is simple, $\ker\pi$ is either $\{e\}$ or A_n. If $\ker\pi = \{e\}$ then π is injective. then $|A_n| \le |\mathrm{Perm}(S_n/A)|$. Thus

$$\frac{n!}{2} \le |S_n/A|! \le (n-1)!,$$

or $n \le 2$. this is a contradiction. If $\ker\pi = A_n$ then π is trivial. Thus $\pi(x) = id$ $\forall x \in A_n$, i.e. $yA = xyA$ $\forall x \in A_n, y \in S_n$. thus

$$y^{-1}xy \in A \quad \forall x \in A_n, y \in S_n$$

We can take $y = e$ to see that $A_n \subset A$. Thus $|S_n/A| \le |S_n/A_n| = 2$, which is a contradiction.

(iv)

$n \geq 5$ again. Thus A_n is simple.

Suppose that A is a subgroup of A_n such that $2 \le |A_n/A| < n$. That's all what we know about A. Thus we need an action on A_n/A to understand more this set. A good candidate for the action is A_n. Define $\pi : A_n \to \mathrm{Perm}(A_n/A)$ to be an action on A_n/A by translation. Then π is a homomorphism. Thus, again $\ker\pi$ is either A_n or $\{e\}$. If $\ker\pi = \{e\}$ then π is injective. Thus $|A_n| \leq |Perm(A_n/A)|$, or

$$\frac{n!}{2} \leq |A_n/A|! \leq (n-1)!,$$

or $n \leq 2$ (contradiction).

If $\ker\pi =$ ~~$\{e\}$~~ A_n then π is a trivial map. Thus every $x \in A_n$ does not translate A_n/A at all, That happens only if $A = A_n$, which means $|A_n/A| = 1$ (contradiction).

10 (2) Prove that there are no simple groups of order 90.

Proof Suppose by contradiction that ~~G is a~~ there exists a simple group G of order 90, Note that $90 = 2\cdot 3^2 \cdot 5$. Then G has a 5-Sylow subgroup, call H. Let S be the family of all 5-Sylow subgroups of G, and let G operate on S by conjugation $x \cdot K = xKx^{-1}$ $\forall x \in G, K \in S$. Then there is actually only one orbit in S because every 5-Sylow subgroup of G is conjugate to H. Therefore we see that the stabilizer of H in G is N_H, which couldn't change H in the operation. Thus, the number of conjugate group to H ($\#S$) is $|G/N_H|$. By Sylow's theorem, $|G/N_H| \equiv 1 \mod 5$.

Moreover, since $H < N_H$, $|G/N_H|$ divides $|G/H| = \frac{90}{5} = 18$.

Thus $|G/N_H|$ is a divisor of 18 which is equivalent to 1 mod 5. Then $|G/N_H| \in \{1, 6\}$. If $|G/N_H| = 1$ then $N_H = G$ and H is normal in G (Contradiction!). Then $|G/N_H| = 6$.

Now let G operate on the family of left ~~cots~~ cosets of H by Conjugation $\pi : G \longrightarrow Perm(G/N_H)$

$$x \longmapsto x : x(yN_H) = xyN_H$$

Then π is a (sort of natural) homomorphism from G to $Perm(G/N_H)$. Then $\ker\pi$ is a normal subgroup of G. There are only two possibility. First, if $\ker\pi = G$ then π is trivial. Then $x(yN_H) = yN_H$ $\forall x, y \in G$. Then $y^{-1}xy \in N_H$ $\forall x, y \in G$. By choosing $y = e$, we have $N_H = G$, which again implies H is normal in G (contradiction). The second possibility is that $\ker\pi = \{e\}$. Then π is injective and

$$G \simeq Im\,\pi < Perm(G/N_H) = S_6$$

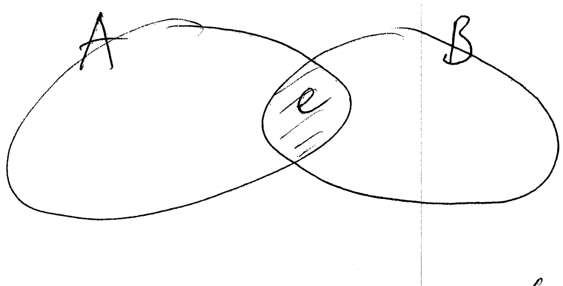
The G can be thought as a subgroup of S_6. By Problem 1, (ii), that G is simple implies $G \subset A_6$. We have

$$|A_6/G| = \frac{|A_6|}{|G|} = \frac{(6!)/2}{90} = 4$$

Thus G is a subgroup of A_6 of index 4. this contradicts problem ①, part (iv).

10 ③ Show that every group of order 231 is the direct product of a group of order 11 and a group of order 21.

Proof Let G be a group of order $231 = 3 \times 7 \times 11$. Then there exists a 11-Sylow subgroup A, 7-Sylow subgroup B, and 3-Sylow subgroup C. The number of 11-Sylow subgroups is $|G/N_A|$, which is $\equiv 1 \bmod 11$. We have $|G/N_A|$ divides $|G/A| = 21$. Thus $|G/N_A| = 1$ and $N_A = G$. Thus A is normal in G. Similarly, the number of 7-Sylow subgroups is $|G/N_B|$, which is $\equiv 1 \bmod 7$. Moreover, $|G/N_B|$ divides $|G/B| = 33$. Thus $|G/N_B| = 1$ and $N_B = G$. Thus B is normal in G. The number of 3-Sylow subgroups is $|G/N_C|$, which is $\equiv 1 \bmod 3$. Moreover, $|G/N_C|$ divides $|G/C| = 77$. Thus $|G/N_C| = 1$ or 7. If $|G/N_C| = 1$ then $N_C = G$ and C is normal in G. If $|G/N_C| = 7$ then there are exactly seven 3-Sylow subgroups of G.



First we will show that $ab = ba$ $\forall a \in A, b \in B$

Since B is normal, A can act on B by conjugate. Then the number of fixed points is

$\equiv |B| \bmod 11$, because A is a 11-group. That means the number of fixed points is exactly 7. Thus every element of B is a fixed point (not being changed under the operation of A). Thus $aba^{-1} = b$ $\forall a \in A, b \in B$, i.e. $ab = ba$ $\forall a \in A, b \in B$.

* <u>If C is normal in G</u>

Then A can act on C by conjugation. ~~These~~ # of fixed points in C is $\equiv |C| \bmod 11$. thus # of fixed points is exactly 3. Thus every element of C is a fixed point, i.e. $aca^{-1} = c$ $\forall c \in C$. Thus $ac = ca$ $\forall a \in A, \forall c \in C$.

Since B is normal in G, BC is a subgroup in G. Indeed, we can actually prove a general statement:

"Let H and K be two subgroups of G and $H \triangleleft G$ (normal in G). Then HK is a subgroup of G."

Proof. ~~Take~~ $h_1, h_2 \in H$, $k_1, k_2 \in K$ arbitrarily. We have

$$\cancel{h_1 h_2 k_1 k_2}\ (h_1 k_1)(h_2 k_2) = \underbrace{h_1}_{\in H}\underbrace{(k_1 h_2 k_1^{-1})}_{\in H}\underbrace{k_1 k_2}_{\in K} \in HK$$

$$(hk)^{-1} = k^{-1}h^{-1} = \cancel{h^{-1} h k}\ \underbrace{k^{-1}h^{-1}k}_{\in H}\underbrace{k^{-1}}_{\in K} \in HK$$

Put $D = BC$, then $|D| = 21$ because $(7,3) = 1$. Since ~~A~~ $ab = ba$ $\forall a \in A, b \in B$ and $ac = ca$ $\forall a \in A, c \in C$, we have $a(bc) = (ab)c = (ba)c = b(ac) = b(ca) = (bc)a$.

Thus ~~A~~ $ad = da$ $\forall a \in A, d \in D$.

Therefore $|AD| = 231 = |G|$ and there is a group isomorphism

$$\phi : A \times D \longrightarrow G$$
$$(a,d) \longmapsto ad$$

Thus $G \simeq A \times D$, a direct product of a group of order 11 and a group of order 21.

* <u>If C is not normal in G</u>

Then $|G/N_C| = 7$, which means $|N_C| = 33$. Thus N_C contains ~~an elem~~ an 11-Sylow subgroup, which must be A since A is the only 11-Sylow subgroup. Thus A can act on C by conjugation. The number of fixed points is $\equiv |C| \bmod 11$. Thus # of fixed points $= 3$, which means every element of C is a fixed point. Thus $ac = ca \ \forall a \in A, c \in C$. Thus, just as the above case, we have $ad = da \ \forall a \in A, d \in D$. And there is a group isomorphism

$$\phi : A \times D \longrightarrow G$$
$$(a,d) \longmapsto ad$$

which concludes the proof.

10 (4) Given an example of a finite group G having p-Sylow subgroups P, Q and R (for some prime p) with $P \cap Q = \{e\}$ and $P \cap R \neq \{e\}$.

Proof We'll show that is the case of $G = A_7$ and $p=3$.

We have $|G| = |A_7| = \frac{7!}{2} = 3.4.5.6.7 = 2^3 . 3^2 . 5.7$. Then a 3-Sylow subgroup of G is any subgroup of order 9. Put

$$P = \langle (123), (456) \rangle$$

$$\cancel{Q = \langle (123)} \quad Q = \langle (234), (567) \rangle$$

$$R = \langle (123), (457) \rangle$$

Then P, Q, R are subgroups of G of 9 elements (the notation (123) means a permutation σ such that $\sigma(1)=2$, $\sigma(2)=3$, $\sigma(3)=1$ and $\sigma(i) = i \;\forall i \neq 1,2,3$).

Moreover, $P \cap Q = \{e\}$ and $(123) \in P \cap R$.

10

(5) Problem 24, Lang p. 77

Let p be a prime number. Show that a group of order p^2 is abelian, and that there are only two such groups up to isomorphism.

I like how you write proofs

Proof Let G be a group of order p^2. Since G is a p-group, it's tempting to operate G on itself by conjugation. The number of fixed points is $\equiv |G| \bmod p$. Since e is alway a fixed point, the # of fixed points is divisible by p. In particular, it is greater than 1. A fixed point in G is actually an element x such that $xy = yx \;\forall y \in G$. Let x be a fixed point, except e, in G.

Then $xy = yx$ $\forall y \in G$. We have $\text{ord}(x) \mid |G|$, or $\text{ord}(x) \mid p^2$. Then $\text{ord}(x)$ is either p or p^2. If $\text{ord}(x) = p^2$ then G is cyclic and $G \simeq \mathbb{Z}_{p^2}$.
If $\text{ord}(x) = p$. Let $y \in G \setminus \langle x \rangle$. Then $\text{ord}(y)$ is either p or p^2. If $\text{ord}(y) = p^2$ then again G is cyclic and $G \simeq \mathbb{Z}_{p^2}$. If $\text{ord}(y) = p$, then $\langle x \rangle \langle y \rangle$ has p^2 element. Thus $\langle x \rangle \langle y \rangle = G$. Since $xy = yx$, the following map is a homomorphism

$$\pi : \langle x \rangle \times \langle y \rangle \longrightarrow G$$
$$(x^i, y^j) \mapsto x^i y^j$$

Since $y \in G \setminus \langle x \rangle$, $\ker \pi = e$ and π is injective. Since $|\langle x \rangle \times \langle y \rangle| = p^2 = |G|$, π is also surjective. Thus π is an isomorphism and $G \simeq \langle x \rangle \times \langle y \rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$.

In short, $G \simeq \mathbb{Z}_{p^2}$ if it is cyclic and $G \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ if it is not.

6 ~~Let G be a group~~ Problem 25, Lang p. 77 10

Let G be a group of order p^3, where p is prime, and G is not abelian. Let Z be its center. Let C be a cyclic group of order p.

(a) Show that $Z \simeq C$ and $G/Z \simeq C \times C$.

(b) Every subgroup of G of order p^2 contains Z and is normal.

(c) Suppose $x^p = 1$ for all $x \in G$. Show that G contains a normal subgroup

$H \simeq C \times C$.

Proof (a) Let G operate on itself by conjugation. Then a fixed point is a element in $Z(G)$ (and vice versa). Since G is a p-group, # fixedpoint is $\equiv |G| \equiv 0 \bmod p$. Thus $|Z(G)| \equiv 0 \bmod p$. Thus $|Z(G)| \in \{p, p^2, p^3\}$.

If $|Z(G)| = p^3$ then $Z(G) = G$ and G is abelian. this is a contradiction.

If $|Z(G)| = p^2$ then $|G/Z(G)| = p$. Then $G/Z(G) = \langle xZ(G) \rangle$ where $x \notin Z(G)$ and $x^p \in Z(G)$. We have $G = Z(G) \cup xZ(G)$. $\forall y, z \in xZ(G)$, we can write $y = xa$, $z = xb$ where $a, b \in Z(G)$. Then

$$yz = (xa)(xb) = xxab = xxba = xbxa = zy.$$

Thus G is abelian. This is also a contradiction.

If $|Z(G)| = p$ then $Z \simeq C$ (we can choose $C = \mathbb{Z}_p$). Then $G/Z(G)$ is a group of order p^2. By the previous problem, $G/Z(G) \simeq \mathbb{Z}_{p^2}$ or $G/Z(G) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$.

If $G/Z(G) \simeq \mathbb{Z}_{p^2}$ then $G/Z(G) = \{Z(G), xZ(G), \ldots, x^{p^2-1}Z(G)\}$, Take $y, z \in G$, we can write $y = x^i a$ and $z = x^j b$ where $0 \le i, j \le p^2 - 1$ and $a, b \in Z(G)$. Then $yz = x^i a x^j b = x^i x^j ab = x^i x^j ba = x^j x^i ba = x^j b x^i a = zy$. Thus G is abelian. This is a contradiction. Thus $G/Z(G) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$.

(b) Let H be a subgroup of G of order p^2. First, we'll show that H is

normal. Let G operate on the family of left cosets of H by translation:
$$\pi : G \longrightarrow \mathrm{Perm}(G/H) \simeq \mathbb{Z}_p$$
$$x \longmapsto x : x.(yH) = xyH$$
Then $x \in \ker \pi$ iff $xyH = yH$ $\forall y \in G$. Thus $x \in \ker\pi$ iff $y^{-1}xy \in H$ $\forall y \in G$. By choosing $y = e$, we have that $x \in \ker\pi$ implies $x \in H$. Thus $\ker \pi \subset H$.

Since $|\ker\pi|$ divides $|G| = p^3$, there are only ~~two~~ 4 possibilities: $|\ker\pi| \in \{1, p, p^2, p^3\}$. Since $|G/\ker\pi|$ divides $|\mathrm{Perm}(G/H)| = p$, $|\ker\pi| \in \{p^2, p^3\}$. If $|\ker\pi| = p^2$ then $|\ker\pi| = |H|$ and $H = \ker\pi$, and H is normal. If $|\ker\pi| = p^3$ then ~~is trivial then~~ then that contradicts $\ker\pi \subset H$. Thus, H is normal.

To show that $Z(G)$ is contained in H, we operate G on H by conjugation (now this is legitimate because H is normal). Then an ~~pt~~ element in H is a fixed point if and only if it is in $Z(G)$. Since e is one fixed point and # of fixed points $\equiv |H| \equiv 0 \mod p$, the number of fixed point is at least p. Since $|Z(G)| = p$, the ~~num~~ fixed points are exactly $Z(G)$. Thus $Z(G) \subset H$.

(c) We know that $Z(G)$ is ~~one~~ of order p. Thus there exists $x \in G \setminus Z(G)$.

By the assumption, x ~~is~~ is of order p. Thus $Z(G)\langle x\rangle = H$ is a subgroup of G, $\underbrace{}_{\text{since } Z(G) \text{ is normal}}$ which has exactly p^2 elements. By part (b), H is normal in G. The map $\pi: Z(G) \times \langle x \rangle \to H$

$$(z, x^i) \mapsto z x^i$$

then determines an isomorphism between $Z(G) \times \langle x \rangle$ and H. Thus

$$H \simeq Z(G) \times \langle x \rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p.$$

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(7) (a) Let G be a group of order pq, where p, q are primes and $p<q$. Assume that $q \not\equiv 1 \mod p$. Prove that G is cyclic.

(b) Show that every group of order 15 is cyclic.

<u>Proof</u> By Sylow's theorem, G has a subgroup of ~~of~~ order p. Let S be the family of all p-subgroup in G. Let $\#S$ is $\equiv 1 \mod p$. Moreover, the $\#S = |G/N_H|$, where H is one p-Sylow subgroup in G. Since $H < N_H$, $|N_H|$ is either p or pq. Thus $\#S = q$ or $\#S = \cancel{pq}\,1$. Since $q \not\equiv 1 \mod p$, the only possibility is that $\#S = 1$. Thus G has only one p-Sylow subgroup H. Thus H is normal. Let K be a q-Sylow subgroup of G. Since H is normal, we can operate K on H by conjugation. Then the # of fixed points $\equiv |H| \mod q$.

Thus # fixed points $\equiv p \mod q$, and it must be p. since $p < q$. Thus every element of H is fixed under the operation of K. Thus $xy = yx$ $\forall x \in H$, $\forall y \in K$.

Moreover, $H \cap K = \{e\}$. Thus there is an isomorphism $\pi : H \times K \to G$
$(x, y) \mapsto xy$

Thus $G \cong H \times K \cong \mathbb{Z}_p \times \mathbb{Z}_q$. Since $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are cyclic with $(p,q) = 1$, $\mathbb{Z}_p \times \mathbb{Z}_q$ is a cyclic group of order pq. Therefore G is cyclic.

(b) we see that $15 = 3 \cdot 5$ and $5 \not\equiv 1 \mod 3$. By part (a), any group of order 15 must be cyclic.

10

(8) Let p, q be distinct primes. Prove that a group of order $p^2 q$ is solvable, and that one of its Sylow subgroups is normal.

Proof Let G be a group of order $p^2 q$ where p and q are distinct primes. Then G has a p-Sylow group H of order p^2, and a q-Sylow subgroup K of order q. If H is normal then we have G is solvable iff H and G/H are solvable. Since H is a p-group, it is solvable. Since $|G/H| = q$, G/H is ~~cyclic~~ abelian and thus solvable. Therefore G is solvable. If H is not normal then the number of p-Sylow subgroup of G is not 1, ~~and $\equiv 1 \mod p$~~. It equals $|G/N_H|$, which divides $|G/H| = q$. Thus it equals q.

~~Let $H_1, \ldots, H_q$ be p-Sylow subgroups of~~ Thus $q \equiv 1 \bmod p$.

In particular, $q > p$. the number of q-Sylow subgroups is $|G/N_K|$, which divides $|G/K| = p^2$. Thus # of q-Sylow subgroups is either 1, p, or p^2. If it is 1 then K is normal; then $|G/K| = p^2$, which is a ~~sub~~ p-group, is ~~also~~ solvable; thus G is solvable. If # of q-Sylow subgroup is p then $p \equiv 1 \bmod q$; this is ~~not~~ impossible because $p < q$. If # of q-Sylow subgroups is p^2, then we call $H_1, H_2, \ldots, H_{p^2}$ to be all of the q-Sylow subgroup of G. Since $|H_i| = q$, $H_i \cap H_j = \{e\}$ for any $i \neq j$.

Thus $\#\left(\bigcup_{i=1}^{p^2} H_i\right) = 1 + (q-1)p^2 = p^2 q + 1 - p^2$. Then $\#\left(G \setminus \bigcup_{i=1}^{p^2} H_i\right) = p^2 - 1$.

Thus the $p^2 - 1$ elements must belong to a p-Sylow subgroup, and ~~that~~ there is only one p-Sylow subgroup. This is a ~~contradi~~ contradiction because we are in the case that there are q p-sylow subgroups.

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(9) (a) Prove that one of the Sylow subgroups of a group of order 40 is normal.

(b) Prove that one of the Sylow subgroups of a group of order 12 is normal.

Proof (a) $40 = 2^3 \times 5$.

Let H be a 5-Sylow subgroup of G, and n be the number of conjugate groups to H. Then $n = |G/N_H|$ which divides $|G/H| = 40/5 = 8$. Moreover, $n \equiv 1 \mod 5$. Thus $n = 1$. Since G has only one 5-Sylow subgroup ~~of~~, H is normal (unchanged under conjugate operation of G).

(b) $12 = 2^2 \times 3$

this case was already consider in the previous problem: either a 2-Sylow subgroup or a 3-Sylow subgroup is normal.