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Math 8202: General Algebra

Homework 1

10/10.

① Problem 1, Lang, p. 165.

Let V be a vector space over a field K, and let U, W be subspaces. We'll show that $\dim U + \dim W = \dim(U+W) + \dim(U \cap W)$. $(*)$

We have a canonical module isomorphism ~~dim~~ $(U+W)/U \simeq W/(W \cap U)$. Thus $\dim (U+W)/U = \dim W/(W \cap U)$. By Theorem 5.3, Lang, p. 144 we have $\dim (U+W)/U + \dim U = \dim(U+W)$, and

$\dim W/(W \cap U) + \dim(W \cap U) = \dim W$.

If $\dim (U+W)/U = \dim W/(W \cap U) = \infty$ then $\dim(U+W) = \dim W = \infty$. Then $(*)$ is true. Otherwise, we have

$$\dim(U+W) - \dim U = \dim (U+W)/U$$

$$\dim W - \dim(W \cap U) = \dim W/(W \cap U)$$

Thus $\dim(U+W) - \dim U = \dim W - \dim(W \cap U)$. Then we get $(*)$.

② Problem 2, Lang, p. 165.

Let R be a nonzero commutative ring, and M be a free module over R. We'll show that M has a well-defined rank. Suppose that M has one basis of n elements and one basis of m elements. Then $M \simeq R^n$ and $M \simeq R^m$ as R-modules.

Because $R \neq \{0\}$ is commutative, ~~there exists~~ it has a maximal ideal $\underline{a}$. Then $R/\underline{a}$ is a field. We have $\underline{a}M$ is a submodule of M and

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$$M/\underline{a}M \simeq (R/\underline{a}) \oplus \cdots \oplus (R/\underline{a}) \quad (n \text{ times})$$
$$\simeq (R/\underline{a})^n.$$

Similarly, $M/\underline{a}M \simeq (R/\underline{a})^m$. Thus $(R/\underline{a})^n \simeq (R/\underline{a})^m$ as R-modules. Thus there exists an isomorphic R-linear map $f : (R/\underline{a})^n \to (R/\underline{a})^m$. Moreover, we know that $(R/\underline{a})^n$ is also an $R/\underline{a}$-module with the multiplication

$$(r+\underline{a})x := rx \quad \forall r \in R,\ x \in (R/\underline{a})^n$$

We'll show that f is also an $R/\underline{a}$-linear map. Indeed

$$f((r+\underline{a})x) = f(rx) = rf(x) = (r+\underline{a})f(x) \quad \forall r \in R,\ x \in R/\underline{a}.$$

Thus f is an isomorphic $R/\underline{a}$-linear map. Since $R/\underline{a}$ is a ~~vector~~ field, $(R/\underline{a})^n$ and $(R/\underline{a})^m$ are vector spaces over $R/\underline{a}$ having dimension n and m respectively. Since they are isomorphic, their dimensions must be equal. Thus, $m = n$.

(3) Let R be an entire ring containing a field k as a subring. Suppose that R is a finite dimensional vector space over k. Then we'll show that R is a field.

We know that every nonzero commutative ring has a maximal ideal. Let $\underline{m}$ be a maximal ideal of R. Then $R/\underline{m}$ is a field. Thus it suffices to show that $\underline{m} = 0$.

Suppose by contradiction that $\underline{m} \neq 0$, i.e. there is $x_0 \in \underline{m} \setminus \{0\}$. We'll show that the set $\{x_0, x_0^2, x_0^3, \ldots\}$ is linearly independent over k. Suppose that there are $c_1, c_2, \ldots, c_\ell \in k$ satisfying $c_1 x_0 + c_2 x_0^2 + \cdots + c_\ell x_0^\ell = 0$. Then

$$x_0 (c_1 + c_2 x_0 + \cdots + c_\ell x_0^{\ell-1}) = 0$$

(or $\forall X$, $\exists c_i$. $c_1 + c_2 X + \cdots + c_n X^n = 0$, $\Rightarrow X \cdot (c_2 + c_3 X + \cdots + c_n X^{n-1}) \frac{1}{-c_1} = 1$ $\Rightarrow X$ invertible.)

Because R is an entire ring, $\underbrace{c_1}_{\in k} + \underbrace{c_2 x_0 + \cdots + c_\ell x_0^{\ell-1}}_{\in \underline{m} \text{ because } \underline{m} \text{ is an ideal}} = 0$.

Thus $c_1 \in k \cap \underline{m}$. Because $\underline{m}$ doesn't contain any invertible element, $c_1 = 0$.

Then we have $c_2 x_0 + \cdots + c_\ell x_0^{\ell-1} = 0$. Similarly, we get

$$x_0 (c_2 + c_3 x_0 + \cdots + c_\ell x_0^{\ell-2}) = 0$$

Then $c_2 + c_3 x_0 + \cdots + c_\ell x_0^{\ell-2} = 0$ because R is entire. Then $c_2 = 0$. By continuing using the above argument, we get $c_1 = c_2 = \cdots = c_\ell = 0$. Thus the set $\{x_0, x_0^2, x_0^3, \ldots\}$ is linearly independent over k. This contradicts the fact that R is a finite dimensional vector space over k.

4 Problem 4(b), Lang, p. 166

Suppose that E and $E_1, E_2, \ldots, E_m$ are modules over a ring R. Let $\varphi_i : E_i \to E$ and $\psi_i : E \to E_i$ be morphisms for each $i = 1, \ldots, m$ such that

$$\sum_{i=1}^m \varphi_i \circ \psi_i = id_E \quad \text{and} \quad \psi_i \circ \varphi_j = \begin{cases} id_{E_j} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Put
$$f : E \longrightarrow E_1 \oplus E_2 \oplus \cdots \oplus E_m$$
$$x \longmapsto (\psi_1(x), \ldots, \psi_m(x))$$

and
$$g : E_1 \times E_2 \times \cdots \times E_m \longrightarrow E$$
$$(x_1, x_2, \ldots, x_m) \longmapsto \sum_{i=1}^m \varphi_i(x_i)$$

We'll show that f and g are module-isomorphisms.

<u>Show that f is a module isomorphism.</u>

- First f is well-defined.
- By the definition of f, it is naturally an R-linear map.

- To show f is injective, we take $x \in \ker f$. Then $\psi_1(x) = \cdots = \psi_m(x) = 0$.

We have by the hypothesis, $x = \sum_{i=1}^m \varphi_i(\psi_i(x)) = \sum_{i=1}^m \varphi_i(0) = 0$.

Thus f is injective.

- To show f is surjective, we take $x_i \in E_i$ arbitrarily for each $i = 1, \ldots, m$.

Define $x = \varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_m(x_m) \in E$. We have

$$\begin{tikzcd} & E_1 \arrow[dl, bend right, "\varphi_1"'] \\ E \arrow[ur, bend right, "\psi_1"'] \arrow[dr, bend left, "\psi_2"] & \\ & E_2 \arrow[ul, "\varphi_2"] \end{tikzcd}$$

$$\psi_i(x) = \psi_i\Big(\sum_j \varphi_j(x_j)\Big) = \sum_j \underbrace{\psi_i \circ \varphi_j(x_j)}_{= \begin{cases} x_i & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}} = x_i.$$

Thus $f(x) = (\psi_1(x), \ldots, \psi_m(x)) = (x_1, \ldots, x_m)$. Thus f is surjective. We've shown that f is a module-isomorphism.

<u>Show that g is a module isomorphism</u>

- First g is well-defined.
- Then g is naturally a module morphism.
- To show that g is injective, we take $(x_1, \ldots, x_m) \in \ker g$. Then $\sum_{i=1}^m \varphi_i(x_i) = 0$.

Then for each $j = 1, \ldots, m$, we have

$$0 = \psi_j(0) = \psi_j\Big(\sum_{i=1}^m \varphi_i(x_i)\Big) = \sum_i \underbrace{\psi_j \circ \varphi_i(x_i)}_{= \begin{cases} x_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}} = x_j$$

Thus $(x_1, \ldots, x_m) = (0, \ldots, 0)$. Thus g is injective.

- To show that g is surjective, we take an arbitrary $x \in E$. For each $i = 1, \ldots, m$, we define $x_i = \psi_i(x)$. Then

$$g(x_1, \ldots, x_m) = \sum_i \varphi_i(x_i) = \sum_i \varphi_i \circ \psi_i(x) = x$$

Thus g is surjective. We've shown that g is a module-isomorphism.

Conversely, now suppose that $E = E_1 \oplus \cdots \oplus E_m$, we can choose to define

$$\varphi_i(x) := (0, \dots, \underset{i}{x}, \dots, 0) \quad \text{for every } x \in E_i \text{, and}$$

$$\psi_i(x_1, x_2, \dots, x_m) := x_i \quad \text{for every } (x_1, \dots, x_m) \in E_1 \oplus \cdots \oplus E_m.$$

Then we get module-morphisms $\varphi_i : E_i \to E$ and $\psi_i : E \to E_i$ for each $i = 1, \dots, m$.

For each $x = (x_1, \dots, x_m) \in E_1 \oplus \cdots \oplus E_m$, we have

$$\varphi_i \circ \psi_i(x) = \varphi_i(\psi_i(x_1, \dots, x_m)) = \varphi_i(x_i) = (0, \dots, \underset{i}{x_i}, \dots 0)$$

Thus $\displaystyle\sum_{i=1}^{m} \varphi_i \circ \psi_i(x) = \sum_{i=1}^{m} (0, \dots, \underset{i}{x_i}, \dots, 0) = (x_1, \dots x_m) = x.$

thus $\displaystyle\sum_{i=1}^{m} \varphi_i \circ \psi_i = id_E$. In addition, for $i, j = 1, \dots, m$, and $t \in E_j$ we have

$$\psi_i \circ \varphi_j(t) = \psi_i(0, \dots, \underset{j}{t}, \dots 0) = \begin{cases} t & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

Thus $\psi_i \circ \varphi_j = id_{E_j}$ if $i = j$ and $\psi_i \circ \varphi_j = 0$ if $i \neq j$.

(5) Problem 14, Lang's, p 169.

From the commutative exact diagram

$$\begin{tikzcd} M' \arrow[r] \arrow[d, "f"'] & M \arrow[r] \arrow[d, "g"'] & M'' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \end{tikzcd}$$

we obtain the following commutative exact diagram by the Snake lemma.

$$\begin{tikzcd}
 & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[dddll, rounded corners, "\delta"] & \\
 & M' \arrow[r] \arrow[d, "f"'] & M \arrow[r] \arrow[d, "g"'] & M'' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N'' \arrow[d] & \\
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h &
\end{tikzcd}$$

We have the exact sequence $\ker f \to \ker g \to \ker h \xrightarrow{\delta} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h$, (*)

where the unnamed morphisms are canonical.

(a) Suppose that f and h are monomorphisms. Then $\ker f \cong \ker h \equiv \{0\}$. Then the first part of the sequence (*) gives $0 \to \ker g \to 0$. Because this sequence is exact, the morphism $\ker g \to 0$ is injective. Thus $\ker g = \{0\}$. Thus g is also a monomorphism.

(b) Suppose that f and h are epimorphisms. Then $\operatorname{coker} f \simeq \operatorname{coker} h = \{0\}$. Then the last part of the sequence (*) gives $0 \to \operatorname{coker} g \to 0$. Because this sequence is exact, the morphism $\operatorname{coker} g \to 0$ is injective. Thus $\operatorname{coker} g = \{0\}$. Thus g is also an epimorphism.

(c) In case both f and h are isomorphism, by part (a) and (b), g is a monomorphism and an epimorphism. Thus g is an isomorphism.

Now assume that we have the exact sequences $0 \to M' \to M$ and $N \to N'' \to 0$. Suppose that f and g are isomorphisms. We'll show that h is also an isomorphism.

$$\begin{tikzcd}
 & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[dddll, bend left, "\delta"] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, "g"] & M'' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r, "\alpha"] \arrow[d, "\varepsilon"] & N'' \arrow[r] \arrow[d, "\beta"] & 0 \\
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r, "\gamma"] & \operatorname{coker} h &
\end{tikzcd}$$

Because f and g are isomorphisms, $\ker f \simeq \ker g \simeq \operatorname{coker} f \simeq \operatorname{coker} g \simeq 0$. Thus the exact sequence (*) is now rewritten as $0 \to 0 \to \ker h \to 0 \to 0 \to \operatorname{coker} h$. Thus $\ker h = 0$. It's left to show $\operatorname{coker} h = 0$. Let $x \in \operatorname{coker} h$. Because β is surjective, there exists $y \in N''$ with $\beta(y) = x$. Because α is surjective, there exists $z \in N$ with $y = \alpha(z)$. Because g is bijective, there exists $t \in M$ with $g(t) = z$. Thus

$$x = \beta(y) = \beta\alpha(z) = \beta\alpha g(t) = \gamma \underbrace{\varepsilon g}_{0}(t) = 0.$$

Thus $\operatorname{coker} h = 0$.

Now suppose that g and h are isomorphisms. We'll show that f is also an isomorphism. The exact sequence (*) with $\ker g \simeq \ker h \simeq \operatorname{coker} g \simeq \operatorname{coker} h \simeq 0$ gives $\ker f \to 0 \to 0 \to \operatorname{coker} f \to 0 \to 0$. Thus $\operatorname{coker} f = 0$. We need to show that $\ker f = 0$. Take $x \in \ker g$. We have $f(x) = 0$. Thus $0 = vf(x) = gu(x)$. Since g is bijective, $u(x) = 0$. Since u is injective, $x = 0$. Thus $\ker f = 0$.

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f"] & M \arrow[d, "g"] \\
 & N' \arrow[r, "v"] & N
\end{tikzcd}$$

⑥ Problem 15, Lang, p. 169

(a) Suppose we have the following commutative ~~exact~~ diagram of R-modules such that each row is exact, and f_1 is surjective, and f_2, f_4 are injective.

$$\begin{tikzcd} M_1 \arrow[r, "\alpha"] \arrow[d, "f_1"'] & M_2 \arrow[r, "\beta"] \arrow[d, "f_2"] & M_3 \arrow[r, "\gamma"] \arrow[d, "f_3"] & M_4 \arrow[d, "f_4"] \\ N_1 \arrow[r, "\delta"] & N_2 \arrow[r, "\varepsilon"] & N_3 \arrow[r, "\rho"] & N_4 \end{tikzcd}$$

We'll show that f_3 is injective. Take $x \in \ker f_3$. We'll show $x = 0$.

$$\begin{tikzcd} t \arrow[r, "\alpha"] \arrow[d, "f_1"'] & y \arrow[r, "\beta"] \arrow[d, "f_2"] & x \arrow[r, "\gamma"] \arrow[d, "f_3"] & 0 \arrow[d, "f_4"] \\ z \arrow[r, "\delta"] & f_2(y) \arrow[r, "\varepsilon"] & 0 \arrow[r, "\rho"] & 0 \end{tikzcd}$$

Then $f_4 \circ \gamma(x) = \rho \circ f_3(x) = \rho(0) = 0$. Since f_4 is injective, $\gamma(x) = 0$. Thus $x \in \ker \gamma = \operatorname{Im} \beta$. Then there is $y \in M_2$ such that $\beta(y) = x$.

Then $\varepsilon \circ f_2(y) = f_3 \circ \beta(y) = f_3(x) = 0$. Thus $f_2(y) \in \ker \varepsilon = \operatorname{Im} \delta$. Then there is $z \in N_1$ with $f_2(y) = \delta(z)$. Because f_1 is surjective, there is $t \in M_1$ with $z = f_1(t)$. Thus $f_2(y) = \delta(z) = \delta \circ f_1(t) = f_2 \circ \alpha(t)$. Thus $f_2(y - \alpha(t)) = 0$. Because f_2 is injective, $y - \alpha(t) = 0$. Thus $y = \alpha(t)$. Then $x = \beta(y) = \beta \circ \alpha(t) = 0$.

(b) Suppose we have the following commutative diagram of R-modules such that each row is exact, and f_5 is injective, and f_2, f_4 are surjective.

$$\begin{tikzcd} M_2 \arrow[r, "\alpha"] \arrow[d, "f_2"'] & M_3 \arrow[r, "\beta"] \arrow[d, "f_3"] & M_4 \arrow[r, "\gamma"] \arrow[d, "f_4"] & M_5 \arrow[d, "f_5"] \\ N_2 \arrow[r, "\delta"] & N_3 \arrow[r, "\varepsilon"] & N_4 \arrow[r, "\rho"] & N_5 \end{tikzcd}$$

We'll show that f_3 is surjective. Take $y \in N_3$. We'll look for $x \in M_3$ such that $f_3(x) = y$.

$$\begin{tikzcd} t \arrow[r, "\beta"] & z \arrow[r, "\gamma"] \arrow[d, "f_4"] & 0 \arrow[d, "f_5"] \\ y \arrow[r, "\varepsilon"] & \varepsilon(y) \arrow[r, "\rho"] & 0 \end{tikzcd}$$

Because f_4 is surjective, there is $z \in M_4$ such that $f_4(z) = \varepsilon(y)$. Then $f_5\gamma(z) = \rho f_4(z) = \rho\varepsilon(y) = 0$. Since f_5 is injective, $\gamma(z) = 0$. Then $z \in \ker\gamma = \operatorname{Im}\beta$. Then there is $t \in M_3$ such that $z = \beta(t)$. Then we have

$$\varepsilon(y) = f_4(z) = f_4\beta(t) = \varepsilon f_3(t).$$

Thus $\varepsilon(y - f_3(t)) = 0$. Then $y - f_3(t) \in \ker\varepsilon = \operatorname{Im}\delta$. Thus there is $u \in N_2$ with $\delta(u) = y - f_3(t)$. Because f_2 is surjective, there is $v \in M_2$ such that $u = f_2(v)$. Then

$$\begin{tikzcd} v \arrow[r, "\alpha"] \arrow[d, "f_2"] & \alpha(v) \arrow[d, "f_3"] \\ u \arrow[r, "\delta"] & y - f_3(t) \end{tikzcd}$$

$$y - f_3(t) = \delta(u) = \delta f_2(v) = f_3\alpha(v).$$

Thus $y = f_3(t) + f_3\alpha(v) = f_3(t + \alpha(v))$.

Therefore we have found $x = t + \alpha(v)$.