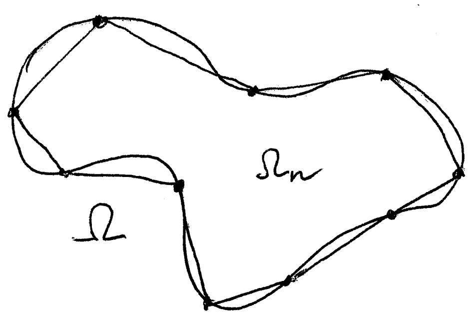
Approximating Riemann maps

<u>Definition</u> Let Ω be an open subset of $\mathbb{C}$. A sequence of open subsets $\{\Omega_n\}$ of $\mathbb{C}$ is said to converge to Ω, written $\lim \Omega_n = \Omega$, if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \{ z \in \mathbb{C}, \operatorname{dist}(z, \Omega^c) > \varepsilon \} \subset \Omega_n \subset \{ z \in \mathbb{C} : \operatorname{dist}(z, \Omega) < \varepsilon \} \quad \forall n > N.$$



<u>Proposition</u> Suppose $\lim \Omega_n = \Omega$. Then for each compact set $K \subset \Omega$, there exists $N \in \mathbb{N}$ such that $K \subset \Omega_n$ for all $n > N$.

<u>Proof</u> Since $K \subset \Omega$, K is compact and Ω is open, there is $\varepsilon > 0$ such that $\operatorname{dist}(z, \Omega^c) > \varepsilon$ for all $z \in K$. By definition, there exists $N \in \mathbb{N}$ such that $\{ z \in \mathbb{C} : \operatorname{dist}(z, \Omega^c) > \varepsilon \} \subset \Omega_n$. Thus $K \subset \Omega_n$ $\forall n > N$. ∎

Suppose that we have a sequence of functions $f_n : \Omega_n \to \mathbb{C}$, and $f : \Omega \to \mathbb{C}$, where $\Omega = \lim \Omega_n$. Because of the above proposition, we can speak of the uniform convergence on every compact set in Ω of (f_n) to f.

<u>Proposition</u> Let Ω be a subset of $\mathbb{C}$, which is

$$\begin{cases} \text{open, connected} \\ \text{simply connected} \\ \text{bounded} \end{cases}$$

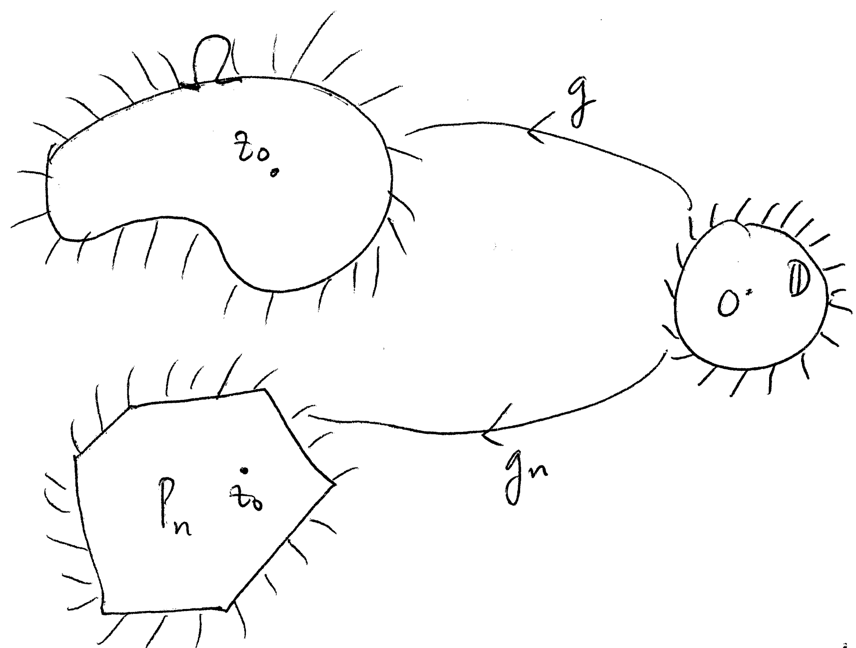
Suppose we have a sequence of open polygons (P_n) with the same 3 properties above such that $\lim P_n = \Omega$. Let $z_0 \in \Omega$ such that $z_0 \in P_n$ for all $n \in \mathbb{N}$. By Riemann mapping theorem, we know that there exists a unique conformal map $f_n : P_n \to \mathbb{D}$ and a conformal map $f : \Omega \to \mathbb{D}$ satisfying the normalization conditions

$$f_n(z_0) = 0, \quad f_n'(z_0) > 0,$$
$$f(z_0) = 0, \quad f'(z_0) > 0.$$

Then (i) (f_n) converges uniformly to f on every compact set in Ω.

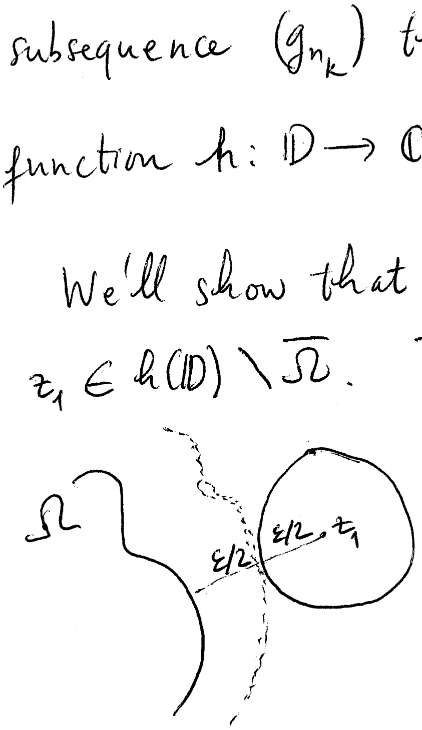
(ii) Denote $g_n : \mathbb{D} \to P_n$, $g : \mathbb{D} \to \Omega$ the inverses of f_n and f respectively. Then (g_n) also converges uniformly to g on every compact set in $\mathbb{D}$.

<u>Proof</u> We'll prove (ii) first, then apply it to prove (i). First, we'll show that (g_n) has a subsequence that converges to g on every compact set in $\mathbb{D}$.



Because Ω is bounded and $\lim P_n = \Omega$, there is a finite ball in $\mathbb{C}$ that contains all P_n for n sufficiently large. Thus the union $\bigcup_{n=1}^{\infty} P_n$ is bounded. This means (g_n) is a sequence of holomorphic functions that are uniformly bounded. By Montel's theorem, there exists a subsequence (g_{n_k}) that converges uniformly on every compact set to a function $h : \mathbb{D} \to \mathbb{C}$. By Weierstrass's theorem, h is also holomorphic.

We'll show that $h(\mathbb{D}) \subset \overline{\Omega}$. Suppose by contradiction that there is $z_1 \in h(\mathbb{D}) \setminus \overline{\Omega}$. Then $\operatorname{dist}(z_1, \Omega) = \varepsilon > 0$. There is $w_1 \in \mathbb{D}$ such that $z_1 = h(w_1)$. There exists $N_1 \in \mathbb{N}$ such that $P_{n_k} \subset \{z : \operatorname{dist}(z, \Omega) < \frac{\varepsilon}{2}\}$ $\forall k > N_1$.



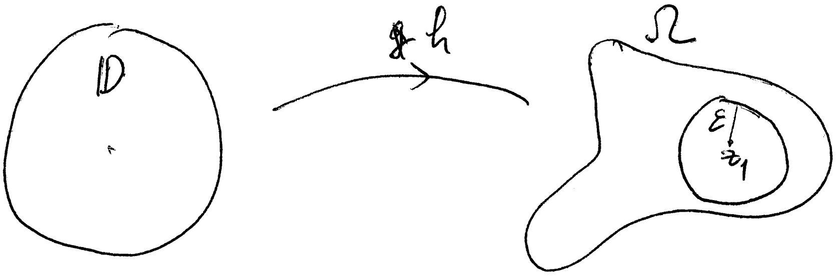
Then $g_{n_k}(w_1) \in P_{n_k}$ $\forall k > N_1$.

Then $\operatorname{dist}(g_{n_k}(w_1), \Omega) < \frac{\varepsilon}{2}$ $\forall k > N_1$.

Letting $k \to \infty$, we get $\mathrm{dist}(\underbrace{h(w_1)}_{z_1}, \Omega) \le \frac{\varepsilon}{2}$. This is a contradiction.

Therefore $h(\mathbb{D}) \subset \overline{\Omega}$. We know that the sequence (g_{n_k}) converges to h on every compact set in $\mathbb{D}$ and each member of the sequence is injective. Thus the limit function h is either injective or constant. Moreover, $h(0) = \lim g_{n_k}(0) = z_0 \in \Omega$. If h is not constant then it's an open map. Since $h(\mathbb{D}) \subset \overline{\Omega}$, we get $h(\mathbb{D}) \subset \Omega$. Therefore, in any case we always have $h(\mathbb{D}) \subset \Omega$.

Next, we'll show that $h(\mathbb{D}) = \Omega$. Suppose by contradiction that this is not true, then there is $z_1 \in \Omega \setminus \underbrace{\overline{h(\mathbb{D})}}_{\text{closure in } \mathbb{C}}$.

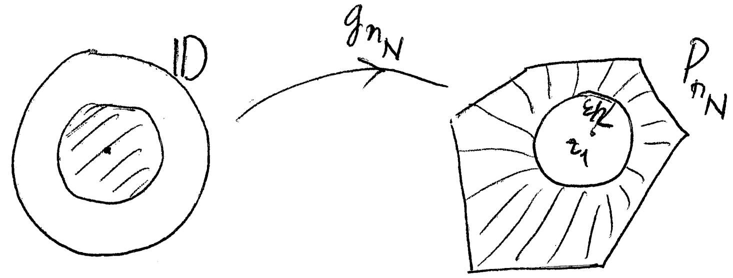


Then there is $\varepsilon > 0$ such that $\overline{B}(z_1, \varepsilon) \subset \Omega \setminus \overline{h(\mathbb{D})}$. Because $\overline{B}(z_1, \varepsilon)$ is a compact subset of Ω, there is $N_1 \in \mathbb{N}$ such that $\overline{B}(z_1, \varepsilon) \subset P_{n_k}$ for all $k \geq N_1$. On the compact subset $\overline{B}(0, \frac{1}{2})$ of $\mathbb{D}$, we know that (g_{n_k}) converges uniformly to h. Thus, there is $N_2 \in \mathbb{N}$ such that

$$|g_{n_k}(w) - h(w)| < \frac{\varepsilon}{2} \quad \forall w \in \overline{B}(0,\tfrac{1}{2}),\ \forall k \geq N_2$$

Put $N = \max(N_1, N_2)$. Then

$$|g_{n_N}(w) - z_1| \geq |h(w) - z_1| - |g_{n_N}(w) - h(w)| > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} \quad \forall w \in \overline{B}(0,\tfrac{1}{2})$$



This means $g_{n_N}(\overline{B}(0,\frac{1}{2})) \subset P_{n_N} \setminus B(z_1, \frac{\varepsilon}{2})$. Since $g_{n_N} : \mathbb{D} \to P_{n_N}$ is conformal, $g_{n_N}(\mathbb{D} \setminus \overline{B}(0,\frac{1}{2})) \subset B(z_1, \frac{\varepsilon}{2})$. However, we know that g_{n_N} can extend to a bijective continuous map from $\overline{\mathbb{D}}$ to $\overline{P_{n_N}}$, which map $\partial\mathbb{D}$ to ∂P_{n_N}. This is a contradiction.

Therefore, $h(\mathbb{D}) = \Omega$. Thus h is not contant. Thus h is injective, and hence a conformal map from $\mathbb{D}$ to Ω. We have $h(0) = z_0$.

By the chain rule,
$$g_n'(w) = \frac{1}{f_n'(g_n(w))}$$

Then
$$g_n'(0) = \frac{1}{f_n'(z_0)} \geq 0$$

Thus $h'(0) = \lim g_n'(0) \geq 0$. Because h is injective, $h'(0) > 0$. Then

$h = g$ by the uniqueness of Riemann maps. Therefore, (g_{n_k}) Converges to g ~~on~~ uniformly on every compact set of $\mathbb{D}$.

Now we'll show that the sequence (g_n) converges to g uniformly on every compact set. Suppose by contradiction that this is not true. Then there is a compact set $K \subset \mathbb{D}$, a number $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ we can find $m > n$ such that $|g_m(z_m) - g(z_m)| \geq \varepsilon$ for some $z_m \in K$. We can rephrase this fact as follow: there exists a subsequence (g_{n_m}) of (g_n) and a sequence (z_m) in K such that $|g_{n_m}(z_m) - g(z_m)| \geq \varepsilon$ for all $m \in \mathbb{N}$. By the first part of our proof, the sequence $(g_{n_m})_m$ should have a subsequence that converges uniformly to g on the compact set K. This is a contradiction.

Next, we'll prove (i). We'll break the proof into 4 steps:

<u>Step 1</u> Using the fact that $g_n : \mathbb{D} \to \mathbb{C}$ and $g : \mathbb{D} \to \mathbb{C}$ are injective, we'll show that for $0 < r < r' < 1$, there exists $N \in \mathbb{N}$ such that

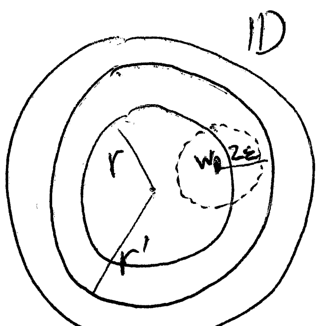
$$g(\overline{B}(0,r)) \subset g_n(B(0,r')) \quad \forall n > N$$

<u>Step 2</u> For every $z \in \Omega$, $f_n(z) \longrightarrow f(z)$.

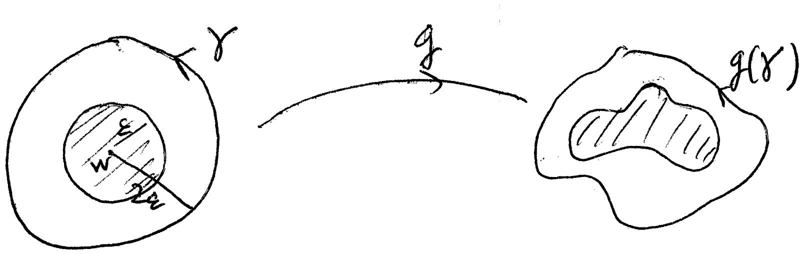
Step 3 We show that the sequence (f_n') is uniformly bounded on every compact set in Ω.

Step 4 We show that (f_n) converges to f uniformly on every compact set in Ω.

Proof of Step 1



For each $w \in \overline{B}(0,r)$, we take a ball of radius $2\varepsilon > 0$ such that $\overline{B}(w, 2\varepsilon) \subset B(0,r')$. Let γ be the circle centered at w, radius 2ε, positively oriented.



Since g is injective, $\underbrace{g(\gamma)}_{\text{compact}} \cap \underbrace{g(\overline{B}(w,\varepsilon))}_{\text{compact}} = \phi$.

Since (g_n) converges to g on every compact set, there is $N_1 \in \mathbb{N}$ such that $g_n(\gamma) \cap g_n(\overline{B}(w,\varepsilon)) = \phi$ for all $n > N_1$. For such n's, we define a map $h_n : B(w,\varepsilon) \to \mathbb{Z}$

$$h_n(\xi) = \frac{1}{2\pi i} \int_\gamma \frac{g_n'(\tau)}{g_n(\tau) - g(\xi)} d\tau$$

Then h_n is well-defined and continuous. Thus h_n is constant.

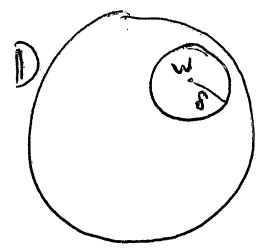
Moreover, we have
$$h_n(w) = \frac{1}{2\pi i}\int_\gamma \frac{g_n'(\tau)}{g_n(\tau)-g(w)}\,d\tau$$
$$\longrightarrow \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{g'(\tau)}{g(\tau)-g(w)}\,d\tau = 0}$$
because the equation $g(\tau) = g(w)$ has only one root in $B(w, 2\varepsilon)$.

Thus there is $N_2 \in \mathbb{N}$ such that $h_n(w) = 1$ for all $n > N_2$. Thus $h_n \equiv 1$ for all $n > N_2$. Thus for each $\xi \in B(w,\varepsilon)$, there exists $\tau \in B(w, 2\varepsilon)$ such that $g_n(\tau) = g(\xi)$. Thus
$$g(B(w,\varepsilon)) \subset g_n(B(w, 2\varepsilon)) \quad \forall n > N_2$$
Since $B(w,2\varepsilon) \subset B(0,r')$, we have $g(B(w,\varepsilon)) \subset g_n(B(0,r'))$ $\forall n > N_2$.

Because ε and N_2 depend on w, we could have denoted them as ε_w and N_w. Since $\{B(w,\varepsilon_w)\}_{w\in \overline{B}(0,r)}$ is an open cover of $\overline{B}(0,r)$, there is an open subcover $B(w_1,\varepsilon_1), \dots, B(w_k,\varepsilon_k)$. If $n > \max\{N_{w_1}, \dots, N_{w_k}\}$, we get $g(B(w_1,\varepsilon_1)), \dots, g(B(w_k,\varepsilon_k)) \subset g_n(B(0,r'))$. Thus,
$$g(\overline{B}(0,r)) \subset g_n(B(0,r')).$$

Proof of Step 2 Take $z \in \Omega$ and put $w = f(z) \in \mathbb{D}$.

Take $\delta > 0$ such that $B(w,\delta) \subset \mathbb{D}$. We define



$$\tilde{g} : \mathbb{D} \to \mathbb{C}, \quad \tilde{g}(\xi) = g(\delta \xi + w),$$

$$\tilde{g}_n : \mathbb{D} \to \mathbb{C}, \quad \tilde{g}_n(\xi) = g_n(\delta \xi + w).$$

Then $\tilde{g}$ and $\tilde{g}_n$ are injective, and $\tilde{g}_n$ converges to $\tilde{g}$ uniformly on every compact set in $\mathbb{D}$. For each $\varepsilon < \frac{1}{2}$, we have $0 < \varepsilon < 2\varepsilon < 1$.

Thus by Step 1, there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\tilde{g}(\overline{B}(0,\varepsilon)) \subset \tilde{g}_n(B(0,2\varepsilon)) \quad \forall n > N(\varepsilon).$$

We have $\tilde{g}(0) = g(w) = z$. Thus $z \in \tilde{g}_n(B(0,2\varepsilon)) = g_n(B(w, 2\varepsilon\delta))$.

Applying f_n both sides, we get $f_n(z) \in B(w, 2\varepsilon\delta)$ for all $n > N(\varepsilon)$.

Thus $f_n(z) \to f(z)$ as $n \to \infty$.

Proof of Step 3

Let $K \subset \Omega$ be a compact set. Then $f(K)$ is a compact set in $\mathbb{D}$.

Thus there is $0 < r < 1$ such that $f(K) \subset B(0,r)$. For $r' = \frac{1+r}{2}$,

we have $0 < r < r' < 1$. By Step 1, there is $N \in \mathbb{N}$ such that

$$g(\overline{B}(0,r)) \subset g_n(B(0,r')) \quad \forall n > N.$$

Then $K = g(f(K)) \subset g(B(0,r)) \subset g_n(B(0,r'))$.

Then $f_n(K) \subset f_n \circ g_n(B(0,r')) = B(0,r'), \quad \forall n > N$.

We know that g' is nonzero on $\overline{B}(0,r')$. Thus $\min\limits_{\overline{B}(0,r')} |g'(w)| = \delta > 0$.

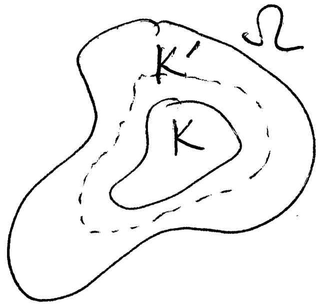
Since (g_n') converges to g' uniformly on $\overline{B}(0,r')$. There is $N_1 \in \mathbb{N}$ such that $|g_n'(w)| \geq \frac{\delta}{2}$ for all $n > N_1$, $w \in \overline{B}(0,r')$.

For every $z \in K$, $\quad |f_n'(z)| = \dfrac{1}{|g_n'(f_n(z))|} \leq \dfrac{2}{\delta} \quad \forall n > N_1, \forall z \in K$.

Thus (f_n') is uniformly bounded on K.

Proof of step 4

Let $K \subset \Omega$ be a compact set. Put $\delta_1 = \frac{1}{3}\operatorname{dist}(K, \Omega^c)$. Then $K' = K + \overline{B}(0,\delta_1) \subset \Omega$. There is $N_1 \in \mathbb{N}$ such that $K' \subset P_n$ for all $n > N_1$. For each n's, we put $h_n(z) = f_n(z) - f(z)$ for all $z \in K'$. Then $h_n'(z) = f_n'(z) - f'(z)$. Since $\{f_n'\}$ is uniformly bounded on K', $\{h_n'\}$ is also uniformly bounded on K'. Put $M = \sup\limits_{\substack{n > N_1 \\ z \in K'}} |h_n'(z)|$.



For each $\varepsilon > 0$, we put $\delta_2 = \min\{\frac{\varepsilon}{2M}, \delta_1\}$.

Then $B(z, \delta_2) \subset P_n$ for all $n > N_1$, and

$B(z, \delta_2) \subset \Omega$. We have

$$h_n(z') = h_n(z) + \int_z^{z'} h_n'(\xi)\, d\xi \qquad \forall z' \in B(z, \delta_2).$$

Thus, $|h_n(z') - h_n(z)| \leq M|z - z'| < M\delta_2 \leq \frac{\varepsilon}{2} \quad \forall n > N_1, z' \in B(z, \delta_2)$.

Because $\{B(z, \delta_2)\}_{z \in K}$ is an open cover of K, we have a finite subcover

$B(z_1, \delta_2), \dots, B(z_m, \delta_2)$ with $z_1, \dots, z_m \in K$. Because of step 2,

$f_n(z_i) \longrightarrow f(z_i)$ as $n \to \infty$. Thus there is $N_2 \in \mathbb{N}$ such that

$$|f_n(z_i) - f(z_i)| < \frac{\varepsilon}{2} \quad \forall i = 1, \dots, m, \ \forall n > N_2.$$

Thus $|h_n(z_i)| < \frac{\varepsilon}{2}$ for such i and ~~m~~. Then

$$|h_n(z)| \leqslant |h_n(z) - h_n(z_i)| + |h_n(z_i)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n > N_2, \cancel{\forall z \in K}.$$
$$\forall z \in B(z_i, \delta_2).$$

Because $K \subset \cup B(z_i, \delta_2)$, we get $|h_n(z)| < \varepsilon \quad \forall n > N_2, \forall z \in K$.

Thus $|f_n(z) - f(z)| < \varepsilon \quad \forall n > N_2, \forall z \in K$.

Therefore (f_n) converges to f uniformly on every compact set in Ω.