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Math 8385: Calculus of Variations

Homework # 1

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① Example 1.2.1, p. 12, Jost - Li Jost

We'll determine all minimizers u of $I(v) = \int_{-1}^{1} (1-(\dot{v})^2)\,dt$ among the set $\{v \in C^{0,1}([-1,1],\mathbb{R}) \mid v(-1) = v(1) = 1\}$.

First we see that $I(v) \geq 0$ for all $v \in C^{0,1}([-1,1])$. Let $u \in C^{0,1}([-1,1])$, $u(-1) = u(1) = 1$ be a minimizer of I. It is sufficient to determine all these u's that satisfy $I(u) = 0$. ~~This is~~ Note that $C^{0,1}([-1,1]) = W^{1,\infty}([-1,1])$ (Rudin, Real and Complex Analysis, p. 157). Thus $\dot{u}$ exists almost everywhere and belongs to $L^\infty([-1,1])$. ✓ Then $I(u) = 0$ if and only if $\dot{u} = \pm 1$ for almost every $t \in [-1,1]$. Because $\dot{u}$ is measurable, the set

$$A = \{t \in [-1,1] \,;\, \dot{u}(t) = 1\}$$

is also measurable. Thus $\dot{u}(t) = \chi_A(t) - \chi_{[-1,1]\setminus A}(t)$ for almost every $t \in [-1,1]$.

Because $u(-1) = 1$, we have

$$u(t) = 1 + \int_{-1}^{t} (\chi_A(s) - \chi_{[-1,1]\setminus A}(s))\,ds \qquad \forall\, t \in [-1,1] \quad (*)$$

Conversely, any function u given by $(*)$ for some measurable subset A of $[-1,1]$ belongs to $C^{0,1}([-1,1]) = W^{1,\infty}([-1,1])$ and makes $I(u) = 0$. Thus, u given by $(*)$ is an admissible minimizer of I if and only if $u(1) = 1$ ✓

$\Leftrightarrow$ A has measure 1

good 10/10

$$u(1)=1 \iff \int_{-1}^{1}\left(\chi_A(s)-\chi_{[-1,1]\setminus A}(s)\right)ds = 0$$

$$\iff |A| - |[-1,1]\setminus A| = 0$$

$$\iff |A| - (2-|A|) = 0$$

$$\iff |A| = 1.$$

In conclusion, all minimizers in $C^{0,1}([-1,1])$ of I that vanish at ± 1 are given by $u(t) = 1 + \int_{-1}^{t}\left(\chi_A(s) - \chi_{[-1,1]\setminus A}(s)\right)ds$ where A is any measurable subset of A such that $|A|=1$.

② Problem 1.1, p. 30, Jost - Li Jost

For mappings $u:[a,b]\to\mathbb{R}^d$, we consider

$$E(u) = \frac{1}{2}\int_a^b |\dot u(t)|^2\,dt,$$

$$L(u) = \int_a^b |\dot u(t)|\,dt.$$

(i) E can be written as $E(u) = \int_a^b F(t, u(t), \dot u(t))\,dt$, where $F:[a,b]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, $F(t,u,p) = \frac{1}{2}|p|^2$.

- Find the Euler-Lagrange equations

We have $F_u(t,u,p) = 0$ and $F_p(t,u,p) = \frac{d}{dp}\left(\frac{1}{2}|p|^2\right)$ (formal notation)

$$= \left(\frac{\partial}{\partial p_i}\left(\frac{p_1^2+\dots+p_d^2}{2}\right)\right)_{1\le i\le d}$$

$$= (p_i)_{1\le i\le d} = p.$$

The Euler-Lagrange equations that a critical point of E must satisfy are $\frac{d}{dt} F_p(t,u,\dot u) - F_u(t,u,\dot u) = 0$, which are equivalent to $\frac{d}{dt}\dot u = 0$.

This is the same as $\ddot u = 0$.

. <u>Compute the second variations</u>

For $\eta \in C_0^\infty([a,b], \mathbb{R}^d)$ and $s \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, we have

$$\frac{d}{ds} E(u+s\eta) = \frac{d}{ds}\int_a^b \frac{1}{2}|u+s\eta|^2 dt$$

$$= \int_a^b \frac{1}{2}\frac{d}{ds}(|u|^2 + 2su\eta + s^2|\eta|^2)dt$$

$$= \int_a^b \frac{1}{2}(2u\eta + 2s|\eta|^2)dt = \int_a^b (u\eta + s|\eta|^2)dt.$$

The second variations are

$$\delta^2 E(u,\eta) = \frac{d^2}{ds^2} E(u+s\eta)\Big|_{s=0} = \frac{d}{ds}\int_a (u\eta + s|\eta|^2)dt\Big|_{s=0}$$

$$= \int_a^b |\eta|^2 dt$$

If we write $\eta = (\eta_1, \dots, \eta_d)$ then $\delta^2 E(u,\eta) = \int_a^b (\eta_1^2 + \dots + \eta_d^2)dt$.

(ii) We'll show that $L(u) \le \sqrt{2(b-a)E(u)}$.

Apply Schwarz-inequality, we have

$$L(u) = \int_a^b 1.|\dot u(t)|dt \le \left(\int_a^b dt\right)^{1/2}\left(\int_a^b |\dot u(t)|^2 dt\right)^{1/2} = \sqrt{b-a}\sqrt{2E(u)}$$

Thus, $L(u) \le \sqrt{2(b-a)E(u)}$.

The general form of Schwarz inequality is $\int_\Omega fg\,dx \le \left(\int_\Omega f^2 dx\right)^{1/2}\left(\int_\Omega g^2 dx\right)^{1/2}$.

If $\int_\Omega f^2 dx \neq 0$ then the equality happens if and only if $g = \alpha f$ almost everywhere for some constant $\alpha \geq 0$. Here we applied this inequality for $f \equiv 1$ and $g \equiv \dot{u}$. Thus the equality $L(u) = \sqrt{2(b-a)E(u)}$ occurs if and only if $\dot{u} \equiv$ const almost everywhere. In order to make sure that the Schwarz inequality is applicable, we need $\dot{u} \in L^2([a,b])$. Thus a reasonable regularity class of u is $W^{1,2}([a,b])$. good 5/5

③ Problem 1.2, p. 31, Jost-Li Jost

We'll determine all minimizers of $I(u) := \int_{-1}^{1} (1-\dot{u})^2 dt$ with $u(1) = u(-1) = 0$.

We can rewrite I as $I(u) = \int_{-1}^{1} F(t, u, \dot{u}) dt$, where

$$F : [-1,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad F(t,u,p) = (1-p)^2.$$

It is obvious that F and F_p are infinitely smooth. Moreover, $F_{pp} = 2 > 0$ for all $t \in [-1,1]$, $u, p \in \mathbb{R}$. Suppose that $u \in AC([-1,1])$, $u(1) = u(-1) = 0$, is a minimizer of I. Then by the regularity theorem 1.2.3, p. 15, Jost, we have $u \in C^2([-1,1])$. The Euler-Lagrange equation that u satisfies is $\frac{d}{dt} F_p(t,u,\dot{u}) = F_u(t,u,\dot{u})$. Because $F_p(t,u,\dot{u}) = -2(1-\dot{u})$, $F_u(t,u,\dot{u}) = 0$, we get $\ddot{u} = 0$. Thus $u(t) = at + b$ $\forall t \in [-1,1]$ for some constants $a, b \in \mathbb{R}$. Because $u(1) = u(-1) = 0$, we get $a = b = 0$. Thus $u \equiv 0$.

This means if $u \in AC([-1,1])$, $u(1) = u(-1) = 0$, is a minimizer of I

then $u \equiv 0$ and $I(v) \geq I(0) = 2 \quad \forall v \in AC([-1,1]),\ v(\pm 1) = 0.$

The rest task is to show that $I(v) \geq 2$ for all $v \in AC([-1,1]), v(\pm 1)=0$.

We have
$$I(v) = \int_{-1}^{1} (1-\dot{v})^2 dt = \int_{-1}^{1} (1 - 2\dot{v} + (\dot{v})^2)\, dt$$
$$= \int_{-1}^{1} dt - 2\int_{-1}^{1} \dot{v}\, dt + \int_{-1}^{1} (\dot{v})^2 dt$$
$$= 2 - 2\underbrace{(v(1) - v(-1))}_{0} + \int_{-1}^{1} (\dot{v})^2 dt$$
$$\geq 2.$$

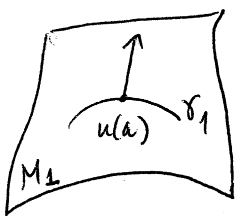
4 Problem 1.3, p.31, Jost-Li Jost.

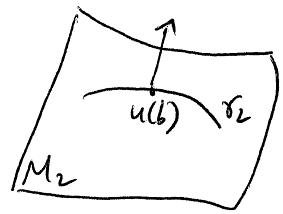
We'll develop a Jacobi field theory for free boundary problems.

Let M_1 and M_2 be two submanifolds of $\mathbb{R}^d$, $F \in C^1([a,b] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ and $u \in C^1([a,b], \mathbb{R}^d)$ be a critical point of $I(u) = \int_a^b F(t, u, \dot{u})\, dt$ with $u(a) \in M_1$ and $u(b) \in M_2$.

First, we'll define the second variations of u. For the case that $u(a)$ and $u(b)$ are fixed, we learned from the introduction part of Section 1.3, page 18 that the second variations of u are given by
$$\delta^2 I(u, \eta) := \frac{d^2}{ds^2} I(u + s\eta)\Big|_{s=0}, \quad \text{where } \eta \in D_0^1([a,b], \mathbb{R}^d).$$

Here we don't fix the values of $u(a)$ and $u(b)$, so η should be allowed to live in a bigger space. We'll analyse the idea of choosing a space for η as follows.





Let $\gamma_1 : [-1,1] \to M_1$ and $\gamma_2 : [-1,1] \to M_2$ be any differentiable curves on M_1 and M_2 such that $\gamma_1(0) = u(a)$ and $\gamma_2(0) = u(b)$. Let $(u_s)_{s\in(-1,1)}$ be a family of functions in $C^1([a,b],\mathbb{R}^d)$ which depend differentiably on s, such that $u_s(a) = \gamma_1(s)$ and $u_s(b) = \gamma_2(s)$, $u_0 = u$. Put $\xi_s(t) = \frac{d}{ds} u_s(t)$.

Then u_s satisfies $u_s(a) \in M_1$ and $u_s(b) \in M_2$. We have

$$\frac{d}{ds} I(u_s) = \int_a^b \frac{d}{ds} F(t, u_s, \dot u_s)\,dt = \int_a^b \left[F_u(t,u_s,\dot u_s)\,\xi_s(t) + F_p(t,u_s,\dot u_s)\,\dot\xi_s(t)\right]dt$$

Thus,
$$\frac{d^2}{ds^2} I(u_s) = \int_a^b \frac{d}{ds}\left[F_u(t,u_s,\dot u_s)\,\xi_s(t)\right]dt + \int_a^b \frac{d}{ds}\left[F_p(t,u_s,\dot u_s)\,\dot\xi_s(t)\right]dt$$

$$= \int_a^b \left[F_{uu}(t,u_s,\dot u_s)\,\xi_s + F_{up}(t,u_s,\dot u_s)\,\dot\xi_s\right]\xi_s\,dt$$

$$+ \int_a^b \left[F_{pu}(t,u_s,\dot u_s)\,\xi_s + F_{pp}(t,u_s,\dot u_s)\,\dot\xi_s\right]\dot\xi_s\,dt$$

$$= \int_a^b \left[F_{uu}(t,u_s,\dot u_s)\,\xi_s\xi_s + 2F_{pu}(t,u_s,\dot u_s)\,\dot\xi_s\xi_s + F_{pp}(t,u_s,\dot u_s)\,\dot\xi_s\dot\xi_s\right]dt \tag{1}$$

Note that the above manipulations are only valid if $F \in C^2$. Put $\xi = \xi_0$.

$$\frac{d^2}{ds^2} I(u_s)\Big|_{s=0} = \int_a^b [F_{uu}(t,u,\dot{u})\zeta\zeta + 2F_{pu}(t,u,\dot{u})\dot{\zeta}\zeta + F_{pp}(t,u,\dot{u})\dot{\zeta}\dot{\zeta}]\,dt$$

Put $\Phi : [a,b] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $\Phi(t,\eta,\xi) = F_{uu}(t,u,\dot{u})\eta\eta + 2F_{pu}(t,u,\dot{u})\xi\eta + F_{pp}(t,u,\dot{u})\xi\xi$.

Then $$\frac{d^2}{ds^2} I(u_s)\Big|_{s=0} = J(\zeta) = \int_a^b \Phi(t,\zeta,\dot{\zeta})\,dt. \qquad (2)$$

This definition has been so far the same as the definition for second variations in the case $u(a)$ and $u(b)$ are fixed. However, there is one difference. We have

$$\zeta(a) = \xi_0(a) = \frac{d}{ds} u_s(a)\Big|_{s=0} = \lim_{s\to 0} \frac{u_s(a) - u_0(a)}{s}$$

$$= \lim_{s\to 0} \frac{\gamma_1(s) - \gamma_1(0)}{s} = \gamma_1'(0).$$

Similarly, $\zeta(b) = \gamma_2'(0)$. Because γ_1 and γ_2 are chosen arbitrarily from the beginning, it is reasonable to say that the definition (2) works for all $\zeta \in D^1([a,b],\mathbb{R}^d)$ which satisfy $\zeta(a) \in T_{u(a)} M_1$ and $\zeta(b) \in T_{u(b)} M_2$. Here $T_{u(a)} M_1$ is the tangent space to M_1 at $u(a)$; similar for $T_{u(b)} M_2$.

This is much more general than the condition $\zeta \in D^1_0([a,b],\mathbb{R}^d)$. Therefore, we come to the definition of second variations for free-boundary problems as follows.

$$\left(\frac{d^2}{ds^2} I(u_s)\Big|_{s=0} =\right) \delta^2 I(u,\zeta) := J(\zeta) = \int_a^b \Phi(t,\zeta,\dot{\zeta})\,dt$$

for all $\zeta \in D^1([a,b],\mathbb{R}^d)$ satisfying $\zeta(a) \in T_{u(a)} M_1$ and $\zeta(b) \in T_{u(b)} M_2$.

If $u \in D^1([a,b],\mathbb{R}^d)$ is a minimizer of I with $u(a) \in M_1$, $u(b) \in M_2$ then $\delta^2 I(u,\zeta) \geq 0$ for all $\zeta \in D^1_u([a,b],\mathbb{R}^d)$, where

$$D^1_u([a,b],\mathbb{R}^d) = \{\zeta \in D^1([a,b],\mathbb{R}^d) : \zeta(a) \in T_{u(a)}M_1,\ \zeta(b) \in T_{u(b)}M_2\}$$

We have obtained an analog of Theorem 1.3.1, p. 19, Jost-Li Jost.

Now we'll give a definition for Jacobi fields. Consider the following accessory problem (where u is a minimizer of I)

$$J(\zeta) = \int_a^b \Phi(t,\zeta,\dot\zeta)\,dt \to \min$$

with the condition $\zeta \in D^1_u([a,b],\mathbb{R}^d)$. We'll say that $\zeta \in C^2([a,b],\mathbb{R}^d)$ is a <u>Jacobi field</u> along $u(t)$ if it satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\Phi_\eta(t,\zeta,\dot\zeta) - \Phi_\xi(t,\zeta,\dot\zeta) = 0 \qquad (3)$$

By the definition of Φ, we have

$$\Phi_\eta(t,\zeta,\dot\zeta) = 2F_{uu}(t,u,\dot u)\zeta + 2F_{pu}(t,u,\dot u)\dot\zeta,$$

$$\Phi_\xi(t,\zeta,\dot\zeta) = 2F_{up}(t,u,\dot u)\zeta + 2F_{pp}(t,u,\dot u)\dot\zeta.$$

Thus (3) is equivalent to

$$\frac{d}{dt}\left[F_{up}(t,u,\dot u)\zeta + F_{pp}(t,u,\dot u)\dot\zeta\right] - F_{uu}(t,u,\dot u)\zeta - F_{pu}(t,u,\dot u)\dot\zeta \qquad (4)$$

which we also call the <u>Jacobi equations</u>.

Now that we have defined the important terms, our goal is to obtain the analogs of Theorems 1.3.2, 1.3.3 and 1.3.4 in Jost, Li-Jost. We emphasize here that the governing equations (the Euler-Lagrange equations for the problems $I \to \min$, $J \to \min$) for the case of fixed-boundary condition and free-boundary condition are the same. The only difference is the space of candidate minimizers. For the problem $I(u) \to \min$, now $u(a)$ and $u(b)$ have more freedom to vary, e.g. $u(a) \in M_1$, $u(b) \in M_2$. For the problem $J(\zeta) \to \min$, now $\zeta(a)$ and $\zeta(b)$ have more freedom to vary, i.e. $\zeta(a) \in T_{u(a)} M_1$, $\zeta(b) \in T_{u(b)} M_2$. We observe that $D_0^1([a,b], \mathbb{R}^d) \subset D_u^1([a,b], \mathbb{R}^d)$. Thus the space of test functions η of the Jacobi equations in fixed-boundary condition case is contained in the space of test functions ζ of the Jacobi equations in free-boundary condition problem. In other words, any $\eta \in D_0^1([a,b], \mathbb{R}^d)$ is also a test function for the latter problem. This makes the Lemma 1.3.1 and Theorem 1.3.2 remain valid (without any necessary generalization). To be more specific, we cite those results in the following.

<u>Lemma 1.3.1'</u>

Let $F \in C^3([a,b] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ and $u \in C^2([a,b], \mathbb{R}^d)$ be a minizer of ~~the problem~~ $I(u) = \int_a^b F(t,u,\dot{u})\,dt$ with $u(a) \in M_1$, $u(b) \in M_2$.

Let $\zeta \in AC([a,b], \mathbb{R}^d)$ be any critical point of $J(\zeta) = \int_a^b \Phi(t,\zeta,\dot{\zeta})dt$. Suppose that $\det(F_{pp}(t,u(t),\dot{u}(t))) \neq 0$ for all $t \in [a,b]$. Then $\zeta \in C^2([a,b],\mathbb{R}^d)$

Proof The proof follows exactly that of Lemma 1.3.1, p. 20. the idea is that $\Phi_{\xi\xi}(t,\zeta,\dot{\zeta}) = 2F_{pp}(t,\zeta,\dot{\zeta})$, whose determinant is nonzero. Then we apply a regularity theorem (Theorem 1.2.3) to conclude $\zeta \in C^2$.

Theorem 1.3.2'

Let $F \in C^2([a,b] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ ~~be~~ of and $u \in C^2([a,b], \mathbb{R}^d)$ be a minimizer of $I(u) = \int_a^b F(t,u,\dot{u})dt$. Then the matrix $F_{pp}(t,u(t),\dot{u}(t))$ is positive-semidefinite for all $t \in [a,b]$.

Proof The proof follows exactly the same as that of Theorem 1.3.2, p. 20.

Theorem 1.3.3'

Let $F \in C^3([a,b] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ and $(u_s)_{s\in(-1,1)}$ be a family of C^2-solutions to the Euler-Lagrange equations

$$\frac{d}{dt}F_p(t,u_s,\dot{u}_s) - F_u(t,u_s,\dot{u}_s) = 0,$$

with $u_s(a) \in M_1$ and $u_s(b) \in M_2$. Suppose that u_s depends differentiably on $s \in (-1,1)$. Then $\zeta(t) := \frac{d}{ds}u_s(t)\Big|_{s=0}$ is a Jacobi field along $u_0(t)$.

Proof The part of the proof showing that ζ satisfies the Euler-Lagrange equations $\frac{d}{dt}\Phi_\xi(t,\zeta,\dot{\zeta}) - \Phi_\eta(t,\zeta,\dot{\zeta}) = 0$ is exactly the same as in the proof of Theorem 1.3.3, page 21. We only need to check if $\xi(a) \in T_{u_0(a)} M_1$ and $\xi(b) \in T_{u_0(b)} M_2$. We have

$$\xi(a) = \frac{d}{ds} u_s(a)\Big|_{s=0} = \lim_{s\to 0} \frac{u_s(a)-u_0(a)}{s} = \lim_{s\to 0} \frac{\gamma_1(s)-\gamma_1(0)}{s} = \gamma_1'(0)$$

where the curve $\gamma_1 : (-1,1) \to M_1$ is defined as $\gamma_1(s) = u_s(a)$. Because $\gamma_1(0) = u_0(a)$, $\xi(a)$ is the tangent vector to γ_1 at $\gamma_1(0) = u_0(a)$. Thus, $\xi(a) \in T_{u_0(a)} M_1$. The same for $\xi(b)$.

Lemma 1.3.2′

Let $F \in C^2([a,b]\times\mathbb{R}^d\times\mathbb{R}^d, \mathbb{R})$ and $u \in C^2([a,b], \mathbb{R}^d)$ be a critical point of $I(u) = \int_a^b F(t,u,\dot u)\,dt$. Let $\xi \in C^2$ be a ~~critical point of~~ Jacobi ~~$J(\xi) = \int_a^b \Phi(t,\xi,\dot\xi)\,dt$~~ field along u on $[a_1, a_2]$ with

$$\Big(F_{pp}(a_1, u(a_1), \dot u(a_1))\,\dot\xi(a_1) + F_{pu}(a_1, u(a_1), \dot u(a_1))\,\xi(a_1)\Big)\,\xi(a_1) = 0$$

and $\xi(a_2) = 0$. Then $\int_{a_1}^{a_2} \Phi(t,\xi,\dot\xi)\,dt = 0$.

Proof Because ξ is a Jacobi field along $u(t)$ on $[a_1,a_2]$, we have

$$\frac{d}{dt}\left[F_{up}(t,u,\dot u)\xi + F_{pp}(t,u,\dot u)\dot\xi\right] - F_{uu}(t,u,\dot u)\xi - F_{pu}(t,u,\dot u)\dot\xi = 0 \quad \forall\, t \in [a_1,a_2].$$

Now multiplying both sides by ξ and taking the integral over $[a_1,a_2]$, we get

$$\int_{a_1}^{a_2} \frac{d}{dt}\left[F_{up}(t,u,\dot u)\xi + F_{pp}(t,u,\dot u)\dot\xi\right]\xi\,dt - \int_{a_1}^{a_2}\left[F_{uu}(t,u,\dot u)\xi + F_{pu}(t,u,\dot u)\dot\xi\right]\xi\,dt = 0 \tag{5}$$

The minuend can be rewritten using integration by part as

$$\underbrace{\left(F_{pp}(t,u,\dot u)\dot\zeta + F_{pu}(t,u,\dot u)\zeta\right)\zeta \Big|_{t=a_1}^{t=a_2}}_{=0} - \int_{a_1}^{a_2} [F_{pp}(t,u,\dot u)\dot\zeta + F_{pu}(t,u,\dot u)\zeta]\dot\zeta\,dt$$

Thus (5) becomes

$$-\int_{a_1}^{a_2} [F_{pp}(t,u,\dot u)\dot\zeta + F_{pu}(t,u,\dot u)\zeta]\dot\zeta\,dt - \int_{a_1}^{a_2} [F_{uu}(t,u,\dot u)\zeta + F_{pu}(t,u,\dot u)\dot\zeta]\zeta\,dt = 0.$$

This is simply $\int_{a_1}^{a_2} \Phi(t,\zeta,\dot\zeta)\,dt = 0.$

Now we are ready to obtain an analog of Theorem 1.3.4, page 22.

Theorem 1.3.4'

Let $F:[a,b]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ be of class C^3 and $u \in C^2([a,b],\mathbb{R}^d)$. Suppose that for all $t\in[a,b]$, the matrix $F_{pp}(t,u(t),\dot u(t))$ is positive-definite. Suppose that there exists $c\in(a,b)$ and $\hat\zeta \in C^2([a,c],\mathbb{R}^d)$, $\hat\zeta \not\equiv 0$, $\hat\zeta(a) \in T_{u(a)}M_1$, $\hat\zeta(c)=0$ such that

$$\left(F_{pp}(a,u(a),\dot u(a))\dot{\hat\zeta}(a) + F_{pu}(a,u(a),\dot u(a))\hat\zeta(a)\right)\hat\zeta(a) = 0, \qquad (6)$$

$$\hat\zeta(c) = 0,$$

$$\frac{d}{dt}\Phi_\xi(t,\hat\zeta,\dot{\hat\zeta}) - \Phi_\eta(t,\hat\zeta,\dot{\hat\zeta}) = 0 \qquad \forall t\in[a,c].$$

Then u is <u>not</u> a minimizer of $I(u)=\int_a^b F(t,u,\dot u)\,dt$ with $u(a)\in M_1$ and $u(b)\in M_2$.

Proof We note that the statement of this problem differs that of Theorem 1.3.4 (which is for fixed-boundary conditions) in that the contraint (6) ~~can~~ replaces $\hat{\xi}(a) = 0$. Here $\hat{\xi}(a) \in T_{u(a)} M_1$ and satisfies (6) instead of being zero.

Suppose by contradiction that u is a ~~minimi~~ minimizer of I in $D^1([a,b], \mathbb{R}^d)$ with $u(a) \in M_1$, $u(b) \in M_2$. Note that by a regularity theorem, $u \in C^2$.

We know that $$J(\zeta) = \int_a^b \Phi(t, \zeta, \dot{\zeta})\,dt \geq 0 \quad \forall \zeta \in D_u^1([a,b), \mathbb{R}^d).$$

Now we define $$\tilde{\xi}(t) = \begin{cases} \hat{\xi}(t), & a \leq t \leq c, \\ 0, & c \leq t \leq b. \end{cases}$$

Then $\tilde{\xi} \in D^1([a,b], \mathbb{R}^d)$. Because $\tilde{\xi}(a) = \hat{\xi}(a) \in T_{u(a)} M_1$ and $\tilde{\xi}(b) = \hat{\xi}(b) = 0 \in T_{u(b)} M_2$, we have $\tilde{\xi} \in D_u^1([a,b], \mathbb{R}^d)$. Thus,

$$J(\tilde{\xi}) = \int_a^b \Phi(t, \tilde{\xi}, \dot{\tilde{\xi}})\,dt \geq 0.$$

Moreover, $$J(\tilde{\xi}) = \int_a^c \Phi(t, \hat{\xi}, \dot{\hat{\xi}})\,dt + \underbrace{\int_c^a \Phi(t, 0, 0)\,dt}_{=0}$$

$$= 0 \text{ (by Lemma 1.3.2')}$$

This means $\tilde{\xi}$ is a minimizer of J. Note that $\tilde{\xi} \in AC([a,b), \mathbb{R}^d)$ and

$$\Phi_{\xi\xi}(t, \tilde{\xi}(t), \xi) = 2 F_{pp}(t, u(t), \dot{u}(t)) \quad \forall t \in [a,b], \xi \in \mathbb{R}^d$$

is a positive definite matrix. Thus, by a regularity theorem, $\tilde{\xi} \in C^2([a,b), \mathbb{R}^d)$

Because $\tilde{\xi}(t)=0$ $\forall t\in(c,b)$, we have $\dot{\tilde{\xi}}(c)=0$. Since $\tilde{\xi}$ is a minimiser of J, it satisfies the E-L equations $\frac{d}{dt}\Phi_{\xi}(t,\tilde{\xi},\dot{\tilde{\xi}})-\Phi_{\eta}(t,\tilde{\xi},\dot{\tilde{\xi}})=0$.

This is Equation (4) which we restate as follows.

$$\frac{d}{dt}\left[F_{pp}(t,u,\dot u)\dot{\tilde{\xi}}+F_{pu}(t,u,\dot u)\tilde{\xi}\right]-F_{uu}(t,u,\dot u)\tilde{\xi}-F_{pu}(t,u,\dot u)\dot{\tilde{\xi}}=0$$

Because every term is regular, we can use the chain rule

$$F_{put}(t,u,\dot u)\tilde{\xi}+F_{puu}(t,u,\dot u)\dot u\tilde{\xi}+F_{pup}(t,u,\dot u)\ddot u\tilde{\xi}+F_{pu}(t,u,\dot u)\dot{\tilde{\xi}}+$$
$$+F_{ppt}(t,u,\dot u)\dot{\tilde{\xi}}+F_{ppu}(t,u,\dot u)\dot u\dot{\tilde{\xi}}+F_{ppp}(t,u,\dot u)\ddot u\dot{\tilde{\xi}}+F_{pp}(t,u,\dot u)\ddot{\tilde{\xi}}$$
$$-F_{uu}(t,u,\dot u)\tilde{\xi}-F_{pu}(t,u,\dot u)\dot{\tilde{\xi}}=0$$

This equation can be rewritten as

$$F_{pp}(t,u,\dot u)\ddot{\tilde{\xi}}+A(t)\dot{\tilde{\xi}}+B(t)\tilde{\xi}=0 \qquad (*)$$

where A and B are continuous functions on $[a,b]$. Because the matrix $F_{pp}(t,u,\dot u)$ is positive definite for all $t\in[a,b]$, it is invertible. By the Cramer's rule of finding inverse matrix, $[F_{pp}(t,u,\dot u)]^{-1}$ is also a continuous function on t. Thus $(*)$ becomes

$$\ddot{\tilde{\xi}}+F_{pp}(t,u,\dot u)^{-1}A(t)\dot{\tilde{\xi}}+F_{pp}(t,u,\dot u)^{-1}B(t)\tilde{\xi}=0$$

This is a homogeneous linear ODE of second order with initial conditions $\tilde{\xi}(c)=\dot{\tilde{\xi}}(c)=0$. Due to the uniqueness of solutions, $\tilde{\xi}\equiv 0$. Thus, $\hat{\xi}\equiv 0$. This is a contradiction.

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