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Math 8385: Calculus of Variations
Homework #2

1) Problem 1.4, Jost–Li Jost, page 31.

For mappings $u : [a,b] \to \mathbb{R}^d$, consider $\displaystyle I(u) = \int_a^b \frac{|\dot{u}(t)|^2}{1+|u(t)|^2}\,dt$

First, we will compute the first variation of I. The integrand is

$$F(t,u,p) = \frac{|p|^2}{1+|u|^2},$$

where $u=(u_1,\dots,u_d)\in\mathbb{R}^d$ and $p=(p_1,\dots,p_d)\in\mathbb{R}^d$. Since F is independent of t, we can write $F(u,p)$ instead of $F(t,u,p)$. For $1\le i\le d$, we have

$$\frac{\partial}{\partial u_i}(|u|^2) = 2u_i \quad \text{and} \quad \frac{\partial}{\partial p_i}(|p|^2) = 2p_i.$$

Thus, $$F_{u_i}(u,p) = |p|^2 \frac{\partial}{\partial u_i}\left(\left(1+|u|^2\right)^{-1}\right) = -|p|^2(1+|u|^2)^{-2}\frac{\partial}{\partial u_i}(|u|^2) = \frac{-2u_i|p|^2}{(1+|u|^2)^2}$$

In form of vector, we have

$$F_u(u,p) = \frac{-2|p|^2 u}{(1+|u|^2)^2} \qquad (1)$$

Also we have $$F_{p_i}(u,p) = \frac{1}{1+|u|^2}\frac{\partial}{\partial p_i}(|p|^2) = \frac{2p_i}{1+|u|^2}.$$

In form of vector, $$F_p(u,p) = \frac{2p}{1+|u|^2} \qquad (2)$$

By definition (page 72, Jost-Li-Jost), the first variation of I is given by

$$\delta I(u,\eta) = \int_a^b \left[F_u(u,\dot u)\cdot \eta + F_p(u,\dot u)\cdot \dot\eta \right] dt,$$

where $\eta \in AC_0([a,b],\mathbb{R}^d)$

Thanks to (1) and (2), we get

$$\delta I(u,\eta) = \int_a^b \left[-\frac{2|\dot u|^2}{(1+|u|^2)^2}(u\cdot\eta) + \frac{2}{1+|u|^2}\,\dot u\cdot\dot\eta \right] dt \qquad (3)$$

Now we'll compute the second variation of I.

For $j\neq i$,

$$F_{u_iu_j}(u,p) = \frac{\partial}{\partial u_j}(F_{u_i}) \overset{(1)}{=\!=} \frac{\partial}{\partial u_j}\left(-\frac{2u_i|p|^2}{(1+|u|^2)^2} \right)$$

$$= -2u_i|p|^2 \frac{\partial}{\partial u_j}\left((1+|u|^2)^{-2} \right)$$

$$= -2u_i|p|^2 \frac{(-2)}{(1+|u|^2)^3} \underbrace{\frac{\partial}{\partial u_j}(|u|^2)}_{2u_j}$$

$$= \frac{8u_iu_j|p|^2}{(1+|u|^2)^3}$$

For $j=i$,

$$F_{u_iu_i}(u,p) = \frac{\partial}{\partial u_i}(F_{u_i}) \overset{(1)}{=\!=} \frac{\partial}{\partial u_i}\left(\frac{-2u_i|p|^2}{(1+|u|^2)^2} \right)$$

$$= -2|p|^2 \frac{\partial}{\partial u_i}\left(u_i(1+|u|^2)^{-2} \right)$$

$$= -2|p|^2\left\{ (1+|u|^2)^{-2} + u_i \frac{\partial}{\partial u_i}\left[(1+|u|^2)^{-2} \right] \right\}$$

$$= -2|p|^2\left[(1+|u|^2)^{-2} - 2u_i(1+|u|^2)^{-3} \underbrace{\frac{\partial}{\partial u_i}(|u|^2)}_{2u_i} \right]$$

$$= -2|p|^2(1+|u|^2)^{-2} + 8u_i^2 \frac{|p|^2}{(1+|u|^2)^3}.$$

Combining the two previous cases, we get

$$F_{u_iu_j}(u,p) = -\frac{2|p|^2}{(1+|u|^2)^2}\delta_{ij} + \frac{8|p|^2}{(1+|u|^2)^3}u_iu_j \qquad (4)$$

where δ_{ij} is the Kronecker delta.

For $1 \le i,j \le d$,

$$F_{u_ip_j}(u,p) = \frac{\partial}{\partial u_i}(F_{p_j}) \overset{(2)}{=\!=} \frac{\partial}{\partial u_i}\left(\frac{2p_j}{1+|u|^2}\right)$$

$$= 2p_j \cdot \frac{-1}{(1+|u|^2)^2} \underbrace{\frac{\partial}{\partial u_i}(|u|^2)}_{2u_i}$$

$$= -\frac{4u_ip_j}{(1+|u|^2)^2} \qquad (5)$$

For $j \ne i$, $F_{p_ip_j}(u,p) = \frac{\partial}{\partial p_j}(F_{p_i}) \overset{(2)}{=\!=} \frac{\partial}{\partial p_j}\left(\frac{2p_i}{1+|u|^2}\right) = 0.$

For $j = i$, $F_{p_ip_i}(u,p) = \frac{\partial}{\partial p_i}(F_{p_i}) \overset{(2)}{=\!=} \frac{\partial}{\partial p_i}\left(\frac{2p_i}{1+|u|^2}\right) = \frac{2}{1+|u|^2}$

Combining two previous cases, ~~the~~ get

$$F_{p_ip_j}(u,p) = \frac{2}{1+|u|^2}\delta_{ij} \qquad (6)$$

By the definition of second variation at Eq. (1.3.6), page 19, Jost-Li Jost we have $\delta^2 I(u,\eta) = \int_a^b \left(F_{u_iu_j}(u,\dot u)\eta_i\eta_j + 2F_{u_ip_j}(u,\dot u)\eta_i\dot\eta_j + F_{p_ip_j}(u,\dot u)\dot\eta_i\dot\eta_j\right)dt$

for $\eta = (\eta_1,\dots,\eta_d) \in AC_0([a,b],\mathbb{R}^d)$.

Thanks to (4), (5), (6) we get

$$\delta^2 I(u,\eta) = \int_a^b \Big\{ \Big[-\frac{2|\dot u|^2}{(1+|u|^2)^2}\delta_{ij} + \frac{8|\dot u|^2}{(1+|u|^2)^3} u_i u_j \Big] \eta_i \eta_j - \frac{8 u_i \dot u_j}{(1+|u|^2)^2} \eta_i \dot\eta_j + \frac{2}{1+|u|^2} \delta_{ij} \dot\eta_i \dot\eta_j \Big\}\, dt$$

$$= \int_a^b \Big[\frac{-2|\dot u|^2}{(1+|u|^2)^2} \delta_{ij} \eta_i \eta_j + \frac{8|\dot u|^2}{(1+|u|^2)^3} (u_i\eta_i)(u_j\eta_j) - \frac{8(u_i\eta_i)(\dot u_j \dot\eta_j)}{(1+|u|^2)^2} + \frac{2}{1+|u|^2} \delta_{ij} \dot\eta_i \dot\eta_j \Big]\, dt$$

In form of index-free,

$$\delta^2 I(u,\eta) = \int_a^b \Big[-\frac{2|\dot u|^2|\eta|^2}{(1+|u|^2)^2} + \frac{8|\dot u|^2 (u\cdot\eta)^2}{(1+|u|^2)^3} - \frac{8(u\cdot\eta)(\dot u\cdot\dot\eta)}{(1+|u|^2)^2} + \frac{2|\dot\eta|^2}{1+|u|^2} \Big]\, dt \tag{7}$$

For each solution u to the Euler-Lagrange equations, the corresponding Jacobi equations are defined by the equations on the top of page 24, Jost-Li Jost. Namely,

$$\frac{d}{dt}\Big(F_{p_i p_j}(u,\dot u)\dot\eta_j + F_{p_i u_j}(u,\dot u)\eta_j \Big) = F_{u_i p_j}(u,\dot u)\dot\eta_j + F_{u_i u_j}(u,\dot u)\eta_j$$

Thanks to (4), (5), (6) we have

$$\frac{d}{dt}\Big(\frac{2}{1+|u|^2}\delta_{ij}\dot\eta_j - \frac{4 u_j \dot u_i}{(1+|u|^2)^2}\eta_j \Big) = \frac{-4 u_i \dot u_j}{(1+|u|^2)^2}\dot\eta_j + \Big(-\frac{2|\dot u|^2}{(1+|u|^2)^2}\delta_{ij} + \frac{8|\dot u|^2}{(1+|u|^2)^3} u_i u_j \Big)\eta_j$$

Using the fact that $\delta_{ij}\dot\eta_j = \dot\eta_i$ and $\delta_{ij}\eta_j = \eta_i$, we have

$$\frac{d}{dt}\left(\frac{2}{1+|u|^2}\dot\eta_i - \frac{4u_j\dot u_i}{(1+|u|^2)^2}\eta_j\right) = \frac{-4u_i\dot u_j}{(1+|u|^2)^2}\dot\eta_j - \frac{2|\dot u|^2}{(1+|u|^2)^2}\eta_i + \frac{8|\dot u|^2}{(1+|u|^2)^3}u_iu_j\eta_j$$

In form of index-free, the Jacobi equations corresponding to u are

$$\frac{d}{dt}\left(\frac{2\dot\eta}{1+|u|^2} - \frac{4\dot u(u\cdot\eta)}{(1+|u|^2)^2}\right) = \frac{-4u(\dot u\cdot\dot\eta)}{(1+|u|^2)^2} - \frac{2|\dot u|^2\eta}{(1+|u|^2)^2} + \frac{8|\dot u|^2(u\cdot\eta)u}{(1+|u|^2)^3} \qquad (8)$$

Before solving for the Jacobi fields η, we'll determine all solutions u to the Euler-Larange equations, subject to the boundary conditions $u(a)=\alpha_1$ and $u(b)=\alpha_2$. By definition, the Euler-Lagrange equations are

$$\frac{d}{dt}\left[F_p(u,\dot u)\right] - F_u(u,\dot u) = 0$$

Thank to (1) and (2), we have

$$\frac{d}{dt}\left(\frac{2\dot u}{1+|u|^2}\right) + \frac{2|\dot u|^2u}{(1+|u|^2)^2} = 0 \qquad (9)$$

Moreover, since the Lagrangian $F=F(u,p)$ is independent of t, we get the first integral by Noether's theorem. Namely,

$$F(u,\dot u) - F_p(u,\dot u)\cdot\dot u = \text{const}$$

Thus, $$\frac{|\dot u|^2}{1+|u|^2} - \frac{2\dot u\cdot\dot u}{1+|u|^2} = \text{const},$$

which is equivalent to $$\frac{|\dot u|^2}{1+|u|^2} = \text{const} \qquad (10)$$

To solve for u in (9), we first consider the case $d=1$, i.e. u is a scalar function. Taking square-root of both sides of (10), we get

$$\frac{\dot{u}}{\sqrt{1+u^2}} = C_1 \qquad (11)$$

Multiplying both sides of (11) by dt and taking indefinite integral both sides, we get
$$\int \frac{du}{\sqrt{u^2+1}} = \int C_1 \, dt.$$

Thus, $\log\left(u+\sqrt{u^2+1}\right) = C_1 t + C_2$. Hence, $u+\sqrt{u^2+1} = \underbrace{\exp(C_1 t + C_2)}_{\alpha(t)}$

We have $u^2+1 = (\alpha(t)-u)^2 = \alpha(t)^2 - 2\alpha(t)u + u^2$. Thus,
$$u = \frac{\alpha(t)^2-1}{2\alpha(t)} = \frac{1}{2}\left(\alpha(t) - \frac{1}{\alpha(t)}\right)$$

Hence,
$$u(t) = \frac{1}{2}\left[\exp(C_1 t + C_2) - \exp(-C_1 t - C_2)\right] \qquad (12)$$

The constants C_1 and C_2 are determined by the boundary conditions $u(a) = \alpha_1$ and $u(b) = \alpha_2$. If $\alpha_1 = \alpha_2$ then by (11), $C_1 = 0$ since there exists some point in (a,b) at which $\dot{u}$ vanishes. Then the constant C_2 is uniquely determined by $\alpha_1 = \alpha_2$. If $\alpha_1 \neq \alpha_2$ then $\dot{u} \not\equiv 0$; then by (11), C_1 is supposed to be nonzero. Then the function on the right hand side of (12) can assume any value in $\mathbb{R}$. In short, given any boundary data α_1, α_2, we can find a unique pair (C_1, C_2) so that the function u given by (12) satisfies $u(a) = \alpha_1$ and $u(b) = \alpha_2$.

Now we double check that the function u given by (12) satisfies the Euler-Lagrange equation (9). Note that if u is given by (12) then it satisfies (11). Thus,

$$(9) \Leftrightarrow \frac{d}{dt}\left(\frac{2C_1}{\sqrt{1+u^2}}\right) + \frac{C_1^2 2u}{1+u^2} = 0$$

$$\Leftrightarrow 2C_1 \frac{d}{dt}\left((1+u^2)^{-1/2}\right) + 2C_1^2 u (1+u^2)^{-1} = 0$$

$$\Leftrightarrow 2C_1\left(-\frac{1}{2}\right)2u\dot{u}(1+u^2)^{-3/2} + 2C_1^2 u (1+u^2)^{-1} = 0$$

$$\Leftrightarrow 2C_1 u (1+u^2)^{-1}\left(\frac{-\dot{u}}{\sqrt{1+u^2}} + C_1\right) = 0,$$

which is true.

Next, we consider the case $d \geq 2$. We notice that the Lagrangian $F(u,p) = \frac{|p|^2}{1+|u|^2}$ is invariant under rotation. More precisely, for any orthogonal matrix $A \in M_d(\mathbb{R})$, we have

$$F(Au, Ap) = \frac{|Ap|^2}{1+|Au|^2} = \frac{|p|^2}{1+|u|^2} = F(u,p).$$

Suppose that $(A_s)_{s\in(-1,1)}$ is a family of orthogonal matrices in $M_d(\mathbb{R})$ which depends differentiably on $s \in (-1,1)$ and $A_0 = I_d$. Then by Noether's theorem, we get the first integral

$$F_p(u,\dot{u}) \cdot \left(\frac{dA_s}{ds}\Big|_{s=0} u\right) = \text{const}.$$

Thanks to (2), we can write $\quad \dfrac{2\dot{u}}{1+|u|^2} \cdot \left(\dfrac{dA_s}{ds}\Big|_{s=0} u\right) = \text{Const} \quad (13)$

For $1 \leq i < j \leq d$, we choose

$$A_s = \begin{array}{cc}
\begin{array}{ccccccccccc} & & & \text{column } i & & & & \text{column } j & & & \end{array} & \\
\begin{pmatrix}
1 & & & & & & & & & & \\
& \ddots & & & & & & & & & \\
& & 1 & & & & & & & & \\
& & & \cos(s) & & & & \sin(s) & & & \\
& & & & 1 & & & & & & \\
& & & & & \ddots & & & & & \\
& & & & & & 1 & & & & \\
& & & -\sin(s) & & & & \cos(s) & & & \\
& & & & & & & & 1 & & \\
& & & & & & & & & \ddots & \\
& & & & & & & & & & 1
\end{pmatrix} & \begin{array}{l} \\ \\ \\ \rightarrow \text{row } i \\ \\ \\ \\ \rightarrow \cancel{\text{column}}\ \text{row } j \\ \\ \\ \\ \end{array}
\end{array}$$

Then each A_s is an orthogonal matrix and $A_0 = I_d$. Moreover,

$$\left.\frac{dA_s}{ds}\right|_{s=0} = \begin{array}{cc}
\begin{array}{cc} \text{column } i & \text{column } j \end{array} & \\
\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} & \begin{array}{l} \rightarrow \text{row } i \\ \rightarrow \text{row } j \end{array}
\end{array}$$

Then

$$\left.\frac{dA_s}{ds}\right|_{s=0} u = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} 0 \\ u_j \\ 0 \\ -u_i \\ 0 \end{pmatrix} & \begin{array}{l} \\ \leftarrow \text{row } i \\ \\ \leftarrow \text{row } j \\ \\ \end{array} \end{array}$$

Then Eq. (12) becomes

$$\frac{2}{1+|u|^2} \begin{pmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_d \end{pmatrix} \cdot \begin{array}{cc} \begin{pmatrix} u_j \\ \\ -u_i \end{pmatrix} & \begin{array}{l} \leftarrow \text{row } i \\ \\ \leftarrow \text{row } j \end{array} \end{array} = \text{const}$$

Equivalently,

$$\frac{2\left(\dot{u}_i u_j - \dot{u}_j u_i\right)}{1+|u|^2} = \text{const} \qquad (14)$$

Therefore, so far we have obtained two kinds of first integrals from the Euler-Lagrange equations (9): one obtained from the time independence, one from the rotation-invariance of the Lagrangian F. They are Eq. (10)

and Eq. (10). We summarize for convenience:

Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{2\dot{u}}{1+|u|^2}\right) + \frac{2|\dot{u}|^2 u}{(1+|u|^2)^2} = 0 \qquad (9)$$

First integral from time-independence:

$$\frac{|\dot{u}|^2}{1+|u|^2} = \text{const} \qquad (10)$$

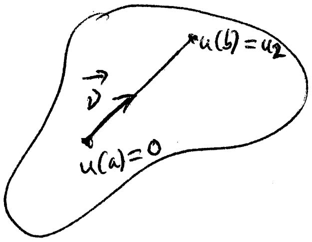
First integrals from rotation-invariance:

$$\frac{\dot{u}_i u_j - \dot{u}_j u_i}{1+|u|^2} = \text{Const} \qquad \forall 1 \leq i < j \leq d. \qquad (14)$$

It is difficult to solve for u from equations (9), (10), (14) because of the nonlinearity. However, in some special cases, the system is soluable. We'll consider two special cases as follows.

- <u>$u(a) = u_1 = 0$ or $u(b) = u_2 = 0$</u>

By plugging $t = a$ or $t = b$ into (14), we get that all constants on the right-hand side of (14) must be zero. Thus $\dot{u}_i u_j - \dot{u}_j u_i = 0$ for all $1 \leq i < j \leq d$. This means the vectors $0, u_i, u_j$ in the space $C^2([a,b], \mathbb{R})$ are colinear. Moreover, since this is true for all indices $1 \leq i < j \leq d$, all vectors $0, u_1, u_2, \ldots, u_d$ in $C^2([a,b], \mathbb{R})$ are colinear. Thus, the vector $u = (u_1, \ldots, u_d)$ in $C^2([a,b], \mathbb{R}^d)$ has constant direction for all $t \in [a,b]$. This direction must be from



from u_1 to u_2. Put $\vec{v} = \dfrac{u_2 - u_1}{|u_2 - u_1|}$. Then $u(t) = f(t)\,\vec{v}$ where $f : [a,b] \to \mathbb{R}$ is a scalar function. Then Eq. (9) becomes

$$\frac{d}{dt}\left(\frac{2\dot{f}\,\vec{v}}{1+f^2}\right) + \frac{2(\dot{f})^2 f\,\vec{v}}{(1+f^2)^2} = 0$$

Equivalently, $\quad \dfrac{d}{dt}\left(\dfrac{2\dot{f}}{1+f^2}\right) + \dfrac{2(\dot{f})^2 f}{(1+f^2)^2} = 0.$

This is exactly the problem in case $d=1$ that we solved earlier. We showed by Eq. (12) that $\quad f(t) = \frac{1}{2}\left[\exp(C_1 t + C_2) - \exp(-C_1 t - C_2)\right].$

The constants C_1 and C_2 are determined by the boundary conditions of f.

If $u(a) = u_1 = 0$ then $f(a) = 0$ and $f(b) = |u_2|$.

If $u(b) = u_2 = 0$ then $f(a) = -|u_1|$ and $f(b) = 0$.

- Find $|u|^2$ in general case

By Eq. (10) and Eq. (14), we can write

$$|\dot{u}|^2 = C\left(1+|u|^2\right), \qquad (14')$$

$$\dot{u}_i u_j - \dot{u}_j u_i = C_{ij}\left(1+|u|^2\right) \quad \forall\, 1 \le i < j \le d, \qquad (14'')$$

where C and C_{ij} are constants. Then

$$\sum_{1\le i<j\le d} (\dot{u}_i u_j - \dot{u}_j u_i)^2 + \left(\sum_{i=1}^{d} \dot{u}_i u_i\right)^2 = \Big(\sum_{1\le i<j\le d} C_{ij}^2\Big)\left(1+|u|^2\right)^2 + \left(\frac{1}{2}\frac{d}{dt}\left(|u|^2\right)\right)^2 \qquad (15)$$

On the other hand,

$$\sum_{1\le i<j\le d} (\dot{u}_i u_j - \dot{u}_j u_i)^2 + \left(\sum_{i=1}^{d} \dot{u}_i u_i\right)^2 = \sum_{1\le i<j\le d} \left[(\dot{u}_i)^2 u_j^2 + (\dot{u}_j)^2 u_i^2 - 2 u_i \dot{u}_i u_j \dot{u}_j\right]$$

$$+ \sum_{i=1}^{d} (\dot{u}_i)^2 u_i^2 + \sum_{1\le i<j\le d} 2 u_i \dot{u}_i u_j \dot{u}_j$$

$$= \sum_{1\le i\ne j\le d} (\dot{u}_i)^2 u_j^2 + \sum_{i=1}^{d} (\dot{u}_i)^2 u_i^2$$

$$= \sum_{i,j=1}^{d} (\dot{u}_i)^2 u_j^2 = \left(\sum_{i=1}^{d} (\dot{u}_i)^2\right)\left(\sum_{j=1}^{d} u_j^2\right)$$

$$= |\dot{u}|^2 |u|^2 \qquad (16)$$

By (15) and (16) we get an identity

$$\left(\sum_{1\le i<j\le d} c_{ij}^2\right)(1+|u|^2)^2 + \left(\frac{1}{2}\frac{d}{dt}(1+|u|^2)\right)^2 = |\dot{u}|^2 |u|^2$$

Put $g(t) = 1 + |u(t)|^2$. Then we have $|\dot{u}|^2 = C g(t)$ by (14'). Then we can rewrite

$$\left(\sum_{1\le i<j\le d} c_{ij}^2\right) g^2 + \left(\frac{1}{2}\frac{dg}{dt}\right)^2 = C g(t)(g(t)-1)$$

Then $$\left(\frac{dg}{dt}\right)^2 = \underbrace{4\left(C - \sum_{1\le i<j\le d} c_{ij}^2\right)}_{\tilde{C}_1} g^2 - \underbrace{4C}_{\tilde{C}_2} g$$

Then $$\frac{dg}{dt} = \sqrt{\tilde{C}_1 g^2 - \tilde{C}_1 g}$$

Then $$\int \frac{dg}{\sqrt{\tilde{C}_1 g^2 - \tilde{C}_2 g}} = \int dt$$

Then $$\frac{1}{\sqrt{2\tilde{C}_1}} \log\left(2\tilde{C}_1 g - \tilde{C}_2 + 2\sqrt{\tilde{C}_1(\tilde{C}_1 g^2 - \tilde{C}_2 g)}\right) = t + \tilde{C}_3$$

Then we get (after a few simple computation steps)

$$g(t) = \frac{\cancel{\alpha t} (\exp(t+\tilde{C}_3) + \tilde{C}_2)^2}{4\tilde{C}_1 \exp(t+\tilde{C}_3)}$$

After renaming the constants, we get $g(t) = \alpha_1 \exp(t) + \alpha_2 + \alpha_3 \exp(-t)$. (17)

Recall that $g(t) = 1+|u|^2$. Thus, we obtained $|u|^2$.

- <u>Solve ~~for~~ the Euler-Lagrange equations (9) in case $d=2$</u>

With the function $g(t) = 1+|u|^2$ given by (17), we can write Eq. (14'') as follows $\dot{u}_i u_j - \dot{u}_j u_i = C_{ij}\, g(t) \quad \forall 1 \leq i < j \leq d$.

In case $d=2$, these equations become $\dot{u}_1 u_2 - \dot{u}_2 u_1 = K g(t)$, where K is a constant. Also, we have

$$\dot{u}_1 u_1 + \dot{u}_2 u_2 = \frac{1}{2}\frac{d}{dt}(1+u_1^2+u_2^2) = \frac{1}{2}\frac{d}{dt}(1+|u|^2) = \frac{1}{2}\frac{dg}{dt}.$$

Thus, we get a system

$$\begin{cases} \dot{u}_1 u_2 - \dot{u}_2 u_1 = K g(t) \\ \dot{u}_1 u_1 + \dot{u}_2 u_2 = \frac{1}{2} g'(t) \end{cases}$$

Thus,
$$\dot{u}_1 = \frac{K g(t) u_2 + \frac{1}{2} g'(t) u_1}{u_1^2+u_2^2} = \frac{K g(t)}{g(t)-1} u_2 + \frac{\frac{1}{2} g'(t)}{g(t)-1} u_1,$$

$$\dot{u}_2 = \frac{\frac{1}{2} g'(t) u_2 - K g(t) u_1}{u_1^2+u_2^2} = \frac{\frac{1}{2} g'(t)}{g(t)-1} u_2 - \frac{K g(t)}{g(t)-1} u_1.$$

In matrix form,

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{g(t)-1} \underbrace{\begin{pmatrix} \frac{1}{2}g'(t) & Kg(t) \\ -Kg(t) & \frac{1}{2}g'(t) \end{pmatrix}}_{A(t)} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad (*)$$

The characteristic polynomial of $A(t)$ is $\det(A(t)-\lambda I) = \left(\frac{1}{2}g'(t)-\lambda\right)^2 + (Kg(t))^2$ which has two distinct roots $\lambda_{1,2}(t) = \frac{1}{2}g'(t) \pm iKg(t)$.

Thus, $A(t)$ is diagonalizable. Then the system $(*)$ is equivalent to a system $\frac{d}{dt}\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \frac{1}{g(t)-1}\begin{pmatrix} \lambda_1(t) & \\ & \lambda_2(t) \end{pmatrix}\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix}$,

where $(\tilde{u}_1, \tilde{u}_2)$ is a linear combination of (u_1, u_2) in $C^2([a,b], \mathbb{R}^2)$.

Then we can solve for $\tilde{u}_1$ and $\tilde{u}_2$ separately because each of them satisfies an ODE of first order.

(2) Problem 2.1, page 60, Jost – Li Jost

Throughout the solution, I will write subscripts instead of superscripts, e.g. (γ_1, γ_2) instead of (γ^1, γ^2), to avoid possible confusion with taking powers.

Denote $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$.

For mappings $\gamma = (\gamma_1, \gamma_2) : \mathbb{R} \to \mathbb{R}^2_+$, we consider

$$E(\gamma) = \frac{1}{2}\int_{\mathbb{R}} \frac{|\dot{\gamma}(t)|^2}{\gamma_2(t)^2}\, dt$$

The integrand is $F(t, \gamma, p) = \frac{1}{2}\frac{|p|^2}{\gamma_2^2}$, where $p = (p_1, p_2) \in \mathbb{R}^2$ and

$\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2_+$. Because F is independent of t, we can write $F(\gamma, p)$ instead of $F(t, \gamma, p)$. We have

$$F_{\gamma_1}(\gamma, p) = 0, \qquad \text{and} \qquad F_{p_1}(\gamma, p) = \frac{p_1}{\gamma_2^2},$$

$$F_{\gamma_2}(\gamma, p) = \frac{-|p|^2}{\gamma_2^3}, \qquad F_{p_2}(\gamma, p) = \frac{p_2}{\gamma_2^2}.$$

The Euler-Lagrange equations are

$$\begin{cases} \dfrac{d}{dt} F_{p_1}(\gamma, \dot\gamma) - F_{\gamma_1}(\gamma, \dot\gamma) = 0, \\ \dfrac{d}{dt} F_{p_2}(\gamma, \dot\gamma) - F_{\gamma_2}(\gamma, \dot\gamma) = 0, \end{cases}$$

which are equivalent to

$$\frac{d}{dt}\left(\frac{\dot\gamma_1}{\gamma_2^2}\right) = 0 \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\dot\gamma_2}{\gamma_2^2}\right) + \frac{|\dot\gamma|^2}{\gamma_2^3} = 0 \qquad (2)$$

From (1) we get $\dfrac{\dot\gamma_1}{\gamma_2^2} = C_1$, where C_1 is a constant. Thus,

$$\dot\gamma_1 = C_1 \gamma_2^2 \qquad (3)$$

Because the Lagrangian F is independent of t, we have a first-integral

$$F(\gamma, \dot\gamma) - F_p(\gamma, \dot\gamma) \cdot \dot\gamma = \text{const},$$

which is equivalent to $\dfrac{1}{2} \dfrac{|\dot\gamma|^2}{\gamma_2^2} - \dfrac{\dot\gamma}{\gamma_2^2} \cdot \dot\gamma = \text{Const}$.

Since $\dot\gamma \cdot \dot\gamma = |\dot\gamma|^2$, we get

$$\frac{|\dot{\gamma}|^2}{\gamma_2^2} = C_2^2 \qquad (4)$$

where $C_2 \geq 0$ is a constant. If $C_2 = 0$ then $|\dot{\gamma}|^2 = 0$. Then $\gamma \equiv$ const, which satisfies (1) and (2). This is the trivial geodesic. Now consider the case $C_2 > 0$. We observe that F is invariant under rescaling. More specifically, we consider a family of transformation $h_s : \mathbb{R}^2 \to \mathbb{R}^2$, $h_s(v) = (1+s)v$ for all $v \in \mathbb{R}^2$, $s \in (-\frac{1}{2}, \frac{1}{2})$. Then $h_0 = id_{\mathbb{R}^2}$ and

$$F(h_s(\gamma), h_s(p)) = F((1+s)\gamma, (1+s)p) = \frac{1}{2} \frac{|(1+s)p|^2}{((1+s)\gamma_2)^2}$$

$$= \frac{1}{2} \frac{|p|^2}{\gamma_2^2} = F(\gamma, p).$$

Therefore, by Noether's theorem, we have

$$F_p(\gamma, \dot{\gamma}) \cdot \frac{d}{ds}\Big|_{s=0} h_s(\gamma) = \text{Const}.$$

Thus, $\quad \dfrac{\dot{\gamma}}{\gamma_2^2} \cdot \dfrac{d}{ds}\Big|_{s=0} (1+s)\gamma = \text{const},$

which is equivalent to $\quad \dfrac{\dot{\gamma} \cdot \gamma}{\gamma_2^2} = C_3, \qquad (5)$

where C_3 is a constant. Thanks to (4), we can rewrite (2) as

$$\frac{d}{dt}\left(\frac{\dot{\gamma}_2}{\gamma_2^2}\right) + \frac{C_2^2}{\gamma_2} = 0$$

$$\Leftrightarrow \ddot{\gamma}_2 \gamma_2^{-2} - 2(\dot{\gamma}_2)^2 \gamma_2^{-3} + C_2^2 \gamma_2^{-1} = 0$$

$$\Leftrightarrow \ddot{\gamma}_2 - 2(\dot{\gamma}_2)^2\gamma_2^{-1} + C_2^2\gamma_2 = 0$$

$$\Leftrightarrow \ddot{\gamma}_2 = \frac{2(\dot{\gamma}_2)^2}{\gamma_2} - C_2^2\gamma_2 \qquad (6)$$

We have

$$(5) \Leftrightarrow \frac{\gamma_1\dot{\gamma}_1 + \gamma_2\dot{\gamma}_2}{\gamma_2^2} = C_3$$

$$\overset{(3)}{\Leftrightarrow} \frac{C_1\gamma_1\gamma_2^2 + \gamma_2\dot{\gamma}_2}{\gamma_2^2} = C_3$$

$$\Leftrightarrow C_1\gamma_1 + \frac{\dot{\gamma}_2}{\gamma_2} = C_3 \qquad (6')$$

- If $C_1 = 0$ then $\frac{\dot{\gamma}_2}{\gamma_2} = C_3$. Thus $\dot{\gamma}_2 - C_3\gamma_2 = 0$. Then $\gamma_2 = C_4 e^{C_3 t}$, where C_4 is a constant. Since $\gamma_2 > 0$, we must have $C_4 > 0$.

 Because $C_1 = 0$, Eq. (3) implies $\dot{\gamma}_1 = 0$. Thus $\gamma_1 \equiv$ const. We conclude that the problem (1), (2) has a family of solutions

$$\gamma(t) = (\alpha_1, \alpha_2 e^{\alpha_3 t}) \qquad \forall t \in \mathbb{R}$$

 where $\alpha_1, \alpha_2, \alpha_3$ are real constants and $\alpha_2 > 0$.

- If $C_1 \neq 0$ then

$$(6') \Leftrightarrow \gamma_1 = \frac{C_3}{C_1} - \frac{1}{C_1}\frac{\dot{\gamma}_2}{\gamma_2}$$

 Taking derivative both sides, we get

$$\dot{\gamma}_1 = -\frac{1}{C_1}\frac{\ddot{\gamma}_2\gamma_2 - (\dot{\gamma}_2)^2}{\gamma_2^2}$$

 Replacing $\dot{\gamma}_1 = C_1\gamma_2^2$ (by Eq. (3)) we get

$$C_1 \gamma_2^2 = -\frac{1}{C_1}\,\frac{\ddot{\gamma}_2 \gamma_2 - (\dot{\gamma}_2)^2}{\gamma_2^2}$$

Thus, $C_1^2 \gamma_2^4 = -\ddot{\gamma}_2 \gamma_2 + (\dot{\gamma}_2)^2$. Dividing both sides by γ_2, we get

$$C_1^2 \gamma_2^3 = -\ddot{\gamma}_2 + \frac{(\dot{\gamma}_2)^2}{\gamma_2}$$

Thus, $$\ddot{\gamma}_2 = \frac{(\dot{\gamma}_2)^2}{\gamma_2} - C_1^2 \gamma_2^3 \qquad (7)$$

Comparing (6) to (7), we get

$$\frac{2(\dot{\gamma}_2)^2}{\gamma_2} - C_2^2 \gamma_2 = \frac{(\dot{\gamma}_2)^2}{\gamma_2} - C_1^2 \gamma_2^3$$

$$\Leftrightarrow \quad \frac{(\dot{\gamma}_2)^2}{\gamma_2} = C_2^2 \gamma_2 - C_1^2 \gamma_2^3$$

$$\Leftrightarrow \quad (\dot{\gamma}_2)^2 = \gamma_2^2 \left(C_2^2 - C_1^2 \gamma_2^2\right)$$

$$\Leftrightarrow \quad \dot{\gamma}_2 = \pm \gamma_2 \sqrt{C_2^2 - C_1^2 \gamma_2^2}$$

Thus, $$\int \frac{d\gamma_2}{\gamma_2 \sqrt{C_2^2 - C_1^2 \gamma_2^2}} = \int \pm dt$$

Hence, $$\log\left(\frac{2C_2^2 + 2C_2\sqrt{C_2^2 - C_1^2 \gamma_2^2}}{\gamma_2} \right) = \pm C_2 t + C_5, \qquad (8)$$

where C_5 is a constant. Because γ_2 is differentiable, the left-hand-side of (8) is differentiable. Thus there is a uniform choice for the plus or minus sign for all $t \in \mathbb{R}$ on RHS(8). Note that the reason why γ_2 is in $C^1(\mathbb{R})$ is as follows. We have $F_{pp}(\gamma, p) = \frac{1}{\gamma_2^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which is positively definite. Then by Theorem 1.2.3, page 15, Jost- Li Jost, any solution $\gamma \in AC([a,b], \mathbb{R}^2)$ of the ~~Euler~~ Euler-Lagrange equations are in class C^2.

If $\gamma_2 : \mathbb{R} \to \mathbb{R}^+$ satisfies (8), it is necessary that $C_2^2 - C_1^2\gamma_2^2 \geq 0$. Thus,

$$0 < \gamma_2 \leq \frac{C_2}{|C_1|}, \quad \forall t \in \mathbb{R}$$

Then we have a lower bound for the expression inside the logarithm on LHS(8). Namely,

$$\frac{2C_2^2 + 2C_2\sqrt{C_2^2 - C_1^2\gamma_2^2}}{\gamma_2} \geq \frac{2C_2^2}{\gamma_2} \geq \frac{2C_2^2}{C_2/|C_1|} = 2C_2|C_1| > 0.$$

Therefore, $LHS(8) \geq \log(2C_2|C_1|) > -\infty$ for all $t \in \mathbb{R}$.

However, with a uniform choice of the plus or minus sign on the right hand side of (8), RHS(8) ranges over all values in $\mathbb{R}$. This is a contradiction.

In conclusion, beside the trivial solution $\gamma \equiv$ const, the Euler-Lagrange equations (1) and (2) has only one family of solutions, namely

$$\gamma(t) = (\alpha_1, \alpha_2 e^{\alpha_3 t}) \quad \forall t \in \mathbb{R}$$

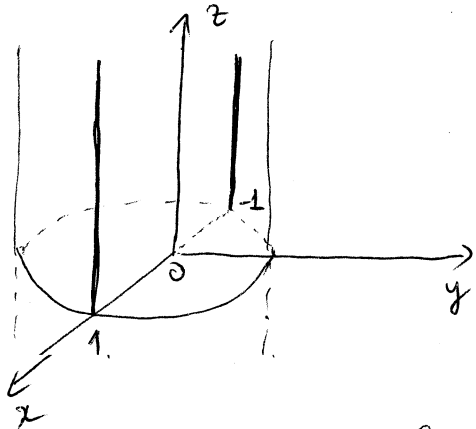
where $\alpha_1, \alpha_2, \alpha_3$ are constants and $\alpha_2 > 0$.

3) Problem 2.3, page 60, Jost- Li Jost.

Consider an infinite cylinder in $\mathbb{R}^3$:

$$M = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$$

Given two points A and B on M, we'll determine the geodesic(s) from A to B on M. First, we'll introduce coordinate charts on M.



Put $\Omega_1 = \{(x,y,z) \in M \mid (x,y) \neq (-1,0)\}$

$\Omega_2 = \{(x,y,z) \in M \mid (x,y) \neq (1,0)\}$.

Then Ω_1 and Ω_2 are open in M and $\Omega_1 \cup \Omega_2 = M$. Define the maps:

$g_1 : \Omega_1 \to \mathbb{R} \times (-\pi,\pi)$, $g_1(x,y,z) = (z,\theta)$, where

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \\ \theta \in (-\pi,\pi) \end{cases}$$

$g_2 : \Omega_2 \to \mathbb{R} \times (0,2\pi)$, $g_2(x,y,z) = (z,\theta)$, where

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \\ \theta \in (0,2\pi) \end{cases}$$

Then g_1 and g_2 are homeomorphisms with the following inverse maps.

$g_1^{-1} = f_1 : \mathbb{R} \times (-\pi,\pi) \to \Omega_1$, $f_1(z,\theta) = (\cos\theta, \sin\theta, z)$

$\forall z \in \mathbb{R}$, $\theta \in (-\pi,\pi)$,

$g_2^{-1} = f_2 : \mathbb{R} \times (0,2\pi) \to \Omega_2$, $f_2(z,\theta) = (\cos\theta, \sin\theta, z)$

$\forall z \in \mathbb{R}$, $\theta \in (0,2\pi)$.

The transition maps are $g_1 \circ g_2^{-1}(z,\theta) = (z, \theta - \pi)$ $\quad \forall z \in \mathbb{R}$, $\theta \in (0,\pi)$,

$g_2 \circ g_1^{-1}(z,\theta) = (z, \theta + \pi)$ $\quad \forall z \in \mathbb{R}$, $\theta \in (0,\pi)$.

Next, we'll compute the metric tensors on each chart.

On chart Ω_1

The coordinate chart is $f_1(z,\theta) = (\underbrace{\cos\theta}_{f_1^1}, \underbrace{\sin\theta}_{f_1^2}, \underbrace{z}_{f_1^3}) \quad \forall z \in \mathbb{R}, \theta \in (-\pi,\pi)$.

By definition, the metric tensor is a 2×2 matrix (g_{ij}) with

$$g_{11} = \frac{\partial f^\alpha}{\partial z}\frac{\partial f^\alpha}{\partial z} = \left(\frac{\partial(\cos\theta)}{\partial z}\right)^2 + \left(\frac{\partial(\sin\theta)}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2 = 1,$$

$$g_{12} = \frac{\partial f^\alpha}{\partial z}\frac{\partial f^\alpha}{\partial \theta} = \frac{\partial(\cos\theta)}{\partial z}\frac{\partial(\cos\theta)}{\partial\theta} + \frac{\partial(\sin\theta)}{\partial z}\frac{\partial(\sin\theta)}{\partial\theta} + \frac{\partial z}{\partial z}\frac{\partial z}{\partial\theta} = 0,$$

$$g_{21} = \frac{\partial f^\alpha}{\partial\theta}\frac{\partial f^\alpha}{\partial z} = g_{12} = 0,$$

$$g_{22} = \frac{\partial f^\alpha}{\partial\theta}\frac{\partial f^\alpha}{\partial\theta} = \left(\frac{\partial(\cos\theta)}{\partial\theta}\right)^2 + \left(\frac{\partial(\sin\theta)}{\partial\theta}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2 = \sin^2\theta + \cos^2\theta = 1.$$

Therefore, $g_{ij} = \delta_{ij}$ for every $(z,\theta) \in \mathbb{R}\times(-\pi,\pi)$.

On chart Ω_2

The expressions for g_{ij} are the same as those in chart Ω_1, except for the range of θ. Namely,

$$g_{ij}(z,\theta) = \delta_{ij} \quad \forall z \in \mathbb{R}, \theta \in (0, 2\pi).$$

Because the metric tensors on each chart are constant matrices, the Christoffel symbols are identically zero. By Lemma 2.1.2, page 39, Jost-Li Jost, the geodesic equations on each coordinate chart of M are given by

$$\ddot{\gamma}(t) + \Gamma^i_{jk}(\gamma(t))\,\dot{\gamma}^j(t)\,\dot{\gamma}^k(t) = 0 \quad \forall i = 1,2,$$

where $\gamma(t) = (\gamma^1(t), \gamma^2(t)) = (z(t), \theta(t))$ and $|\dot{\gamma}| = 1$.

Since $\Gamma^i_{jk} \equiv 0$, we get 3 simple equations

$$\begin{cases} \ddot{z}(t) = 0 \\ \ddot{\theta}(t) = 0 \qquad\qquad \forall t \in [0, L] \\ \dot{z}(t)^2 + \dot{\theta}(t)^2 = 1 \end{cases}$$

where L is the length of the geodesic. Therefore,

$$\begin{cases} z(t) = at + b, & \forall t \in [0, L] \\ \theta(t) = ct + d, & \forall t \in [0, L] \qquad (*) \\ a^2 + c^2 = 1 & (1) \end{cases}$$

To determine a, b, c, d we need some boundary data.

Given two points A and B on M, we have 4 following cases:

(i) $A \in \Omega_1$, $B \in \Omega_1$,

(ii) $A \in \Omega_1$, $B \in \Omega_2$,

(iii) $A \in \Omega_2$, $B \in \Omega_1$,

(iv) $A \in \Omega_2$, $B \in \Omega_2$.

<u>Case (i)</u> $A \in \Omega_1$, $B \in \Omega_1$

In the coordinate chart $\mathbb{R} \times (-\pi, \pi)$, we write $A(z_A, \theta_A)$ and $B(z_B, \theta_B)$.

We'll determine all geodesics from A to B which lie entirely in Ω_1.

The system $(*)$ evaluated at $t = 0$ and $t = L$ gives

$$\begin{cases} z_A = b \\ \theta_A = d \\ a^2 + c^2 = 1 \end{cases} \qquad \text{and} \qquad \begin{cases} z_B = aL + b & (2) \\ \theta_B = L + d & (3) \end{cases}$$

From (2) and (3) we get $a = \dfrac{z_B - z_A}{L}$, $c = \dfrac{\theta_B - \theta_A}{L}$ (4)

Replacing a and c into (1), we get

$$\left(\frac{z_B - z_A}{L}\right)^2 + \left(\frac{\theta_B - \theta_A}{L}\right)^2 = 1$$

Thus, $L = \sqrt{(z_B - z_A)^2 + (\theta_B - \theta_A)^2}$. Therefore, we get the geodesic in Ω_1:

$$\gamma(t) = (z(t), \theta(t)), \text{ with } \begin{cases} z(t) = at + z_A \\ \theta(t) = ct + \theta_A \end{cases} \quad \forall t \in [0, L]$$

and a, c being given by (4).

<u>Case (ii)</u> $A \in \Omega_1$, $B \in \Omega_2$

In the coordinate chart $\mathbb{R} \times (-\pi, \pi)$, we write $A(z_A, \theta_A)$.

In the coordinate chart $\mathbb{R} \times (0, 2\pi)$, we write $B(z_B, \theta_B)$.

We'll determine all geodesics from A to B which depart from Ω_1 and end up in Ω_2 without moving out and in more than once.

At time t, a point in Ω_1 on the geodesic curve has coordinates

$$\begin{cases} z_1(t) = a_1 t + b_1 \\ \theta_1(t) = c_1 t + d_1 \\ a_1^2 + c_1^2 = 1 \end{cases} \qquad (5)$$

At time t, a point in Ω_2 on the geodesic curve has coordinates

$$\begin{cases} z_2(t) = a_2 t + b_2 \\ \theta_2(t) = c_2 t + d_2 \\ a_2^2 + c_2^2 = 1 \end{cases} \qquad (6)$$

Thanks to the transition map $f_1^{-1} \circ f_2$, we have $z_2 = z_1$ and $\theta_2 = \theta_1 + \pi$.

Thus, $a_1 = a_2$, $c_1 = c_2$, $b_1 = b_2$ and $d_2 = d_1 + \pi$. At time $t = 0$, (5) gives

$$\begin{cases} z_A = b_1 \\ \theta_A = d_1 \\ a_1^2 + c_1^2 = 1 \qquad (7) \end{cases}$$

At time $t = L$, (6) gives us $\begin{cases} z_B = a_1 L + b_1, \\ \theta_B = c_1 L + (d_1 + \pi). \end{cases}$

Hence, $a_1 = \dfrac{z_B - z_A}{L}$ and $c_1 = \dfrac{\theta_B - \theta_A - \pi}{L}$.

Replacing a_1 and c_1 into (7), we get $\left(\dfrac{z_B - z_A}{L}\right)^2 + \left(\dfrac{\theta_B - \theta_A - \pi}{L}\right)^2 = 1$.

Therefore, $L = \sqrt{(z_B - z_A)^2 + (\theta_B - \theta_A - \pi)^2}$.

Case 3 $\quad A \in \Omega_2$, $B \in \Omega_1$

In chart Ω_2, we write $A(z_A, \theta_A)$. In chart Ω_1, $B(z_B, \theta_B)$.

At time t, a point on the geodesic curve in Ω_2 has coordinates

$$\begin{cases} z_2(t) = a_2 t + b_2 \\ \theta_2(t) = c_2 t + d_2 \\ a_2^2 + c_2^2 = 1 \end{cases}$$

At time t, a point on the geodesic curve in Ω_1 has coordinates

$$\begin{cases} z_1(t) = a_1 t + b_1 \\ \theta_1(t) = c_1 t + d_1 \\ a_1^2 + c_1^2 = 1 \end{cases}$$

Then by the similar arguments to the previous case, we get

$$b_1 = b_2 = z_A, \quad d_2 = \theta_A, \quad d_1 = d_2 - \pi = \theta_A - \pi,$$

$$a_1 = a_2 = \frac{z_B - z_A}{L}, \quad c_1 = c_2 = \frac{\theta_B - \theta_A + \pi}{L},$$

$$L = \sqrt{(z_B - z_A)^2 + (\theta_B - \theta_A + \pi)^2}$$

Case (iv) $A \in \Omega_2$, $B \in \Omega_2$

In the coordinate chart $\mathbb{R} \times (0, 2\pi)$, we write $A(z_A, \theta_A)$ and $B(z_B, \theta_B)$.

We'll determine all geodesics from A to B which lie entirely in Ω_2.

Then we repeat exactly the same arguments made in Case (i). We get $\gamma(t) = (z(t), \theta(t))$ with

$$\begin{cases} z(t) = at + z_A, \\ \theta(t) = ct + \theta_A, \end{cases} \quad t \in [0, L]$$

$$L = \sqrt{(z_B - z_A)^2 + (\theta_B - \theta_A)^2}, \quad a = \frac{z_B - z_A}{L}, \quad c = \frac{\theta_B - \theta_A}{L}.$$

(4) Problem 2.2, page 60, Jost - Li Jost.

I would like to change the notations in this problem so that they look more familiar to me. Specifically, the curves $\gamma(t) = (\gamma^1, \dots, \gamma^d)$ shall be written as $u(t) = (u_1(t), \dots, u_d(t))$ and the energy functional shall be $I(u)$ instead of $E(\gamma)$.

Denote $B_d(0,1) := \{x \in \mathbb{R}^d \mid |x| < 1\}$. For curves $u(t) = (u_1, \dots, u_d) : \mathbb{R} \to B_d(0,1)$ consider

$$I(u) = \frac{1}{2} \int_{\mathbb{R}} \frac{|\dot{u}(t)|^2}{(1 - |u(t)|^2)^2} \, dt$$

We'll compute the Euler-Lagrange equations and compute all solutions.

The integrand is $F(t,u,p) = \frac{1}{2}\frac{|p|^2}{(1-|u|^2)^2}$, where $p=(p_1,\dots,p_d)\in\mathbb{R}^d$ and $u=(u_1,\dots,u_d)\in B_d(0,1)$.

Since F is independente of t, we can write $F(u,p)$ instead of $F(t,u,p)$. We have

$$F_{u_i}(u,p) = \frac{1}{2}|p|^2\frac{\partial}{\partial u_i}\left[(1-|u|^2)^{-2}\right] = \frac{|p|^2}{2}(-2)(-1)(1-|u|^2)^{-3}\frac{\partial}{\partial u_i}(|u|^2)$$
$$= |p|^2(1-|u|^2)^{-3}2u_i$$

Thus, $$F_u(u,p) = \frac{2|p|^2u}{(1-|u|^2)^3} \qquad (1)$$

We have $F_{p_i}(u,p) = \frac{p_i}{(1-|u|^2)^2}$. Thus $$F_p(u,p) = \frac{p}{(1-|u|^2)^2} \qquad (2)$$

The Euler-Lagrange equations are $\frac{d}{dt}F_p(u,\dot u) - F_u(u,\dot u) = 0$.

Thanks to (1) and (2), we get

$$\frac{d}{dt}\left(\frac{\dot u}{(1-|u|^2)^2}\right) - \frac{2|\dot u|^2u}{(1-|u|^2)^3} = 0 \qquad (3)$$

Because the Lagrangian $F(u,p)$ is independent of t, we get a first integral $F(u,\dot u) - F_p(u,\dot u)\cdot\dot u = \text{const}$.

Thus, $$\frac{1}{2}\frac{|\dot u|^2}{(1-|u|^2)^2} - \frac{\dot u}{(1-|u|^2)^2}\cdot\dot u = \text{const}$$

With $\dot u\cdot\dot u = |\dot u|^2$, we get $$\frac{|\dot u|^2}{(1-|u|^2)^2} = c_1^2 \qquad (4)$$

where C_1 is a constant. Now we'll consider the case $d=1$. In this case, Eq. (4) becomes

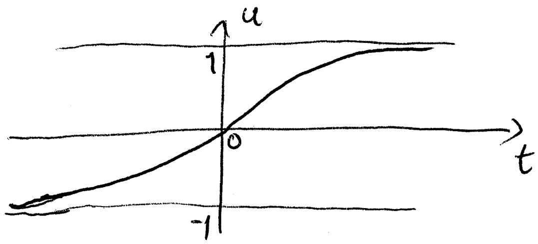
$$\frac{\dot{u}}{1-u^2} = C_1 \qquad (4')$$

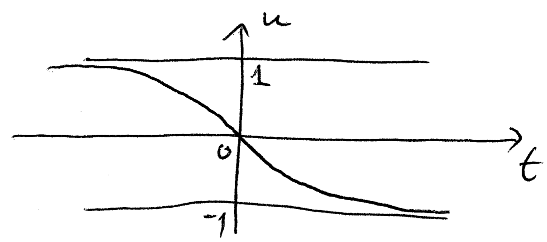
Multiplying both sides by dt and taking indefinite integrals, we get

$$\int \frac{du}{1-u^2} = \int C_1\, dt$$

Thus, $\quad \operatorname{arctanh}(u) = C_1 t + C_2$

Hence, $\quad u(t) = \tanh(C_1 t + C_2) \qquad (4'')$





We'll double-check that u given by (4'') actually satisfies (3). With u given by (4''), it satisfie (4'). Then (3) is equivalent to

$$\frac{d}{dt}\left(\frac{C_1}{1-u^2}\right) - \frac{2C_1^2 u}{1-u^2} = 0$$

$$\Leftrightarrow \quad \frac{2C_1 u \dot{u}}{(1-u^2)^2} - \frac{2C_1^2 u}{1-u^2} = 0$$

$$\Leftrightarrow \quad \frac{2C_1 u}{(1-(u)^2)^2}\left(\frac{\dot{u}}{1-u^2} - C_1\right) = 0, \text{ which is true.}$$

Now consider the case $d \geqslant 2$. If $C_1 = 0$ then (4) implies $\dot{u} \equiv 0$. Then $u(t) \equiv \text{const}$, which is a trivial solution to the Euler-Lagrange equations (3).

If $C_1 \neq 0$, we can assume $C_1 > 0$. We'll follow the same arguments as in page 7 to derive a lot of first integrals from the rotation-inverreant property of F. Specifically, for any orthogonal matrix $A \in M_d(\mathbb{R})$, we have

$$F(Au, Ap) = \frac{1}{2} \frac{|Ap|^2}{(1-|Au|^2)^2} = \frac{1}{2} \frac{|p|^2}{(1-|u|^2)^2} = F(u,p).$$

Suppose that $(A_s)_{s\in(-1,1)}$ is a family of orthogonal matrices in $M_d(\mathbb{R})$ which depends differentiably on $s \in (-1,1)$ and $A_0 = I_d$. Then by Noether's theorem, we get the first integral

$$F_p(u,\dot u) \cdot \left(\frac{dA_s}{ds}\Big|_{s=0} u \right) = \text{Const}$$

Thanks to (2), we can write

$$\frac{\dot u}{(1-|u|^2)^2} \cdot \left(\frac{dA_s}{ds}\Big|_{s=0} u \right) = \text{Const} \qquad (5)$$

For $1 \le i < j \le d$, we choose the family (A_s) as on the top of page 8. Then we still get

$$\frac{dA_s}{ds}\Big|_{s=0} u = \begin{array}{cl} \left(\begin{matrix} 0 \\ u_j \\ 0 \\ -u_i \\ 0 \end{matrix} \right) & \begin{matrix} \\ \leftarrow \text{row } i \\ \\ \leftarrow \text{row } j \\ \\ \end{matrix} \end{array}$$

Then Eq. (5) becomes
$$\frac{1}{(1-|u|^2)^2} \begin{pmatrix} \dot u_1 \\ \vdots \\ \dot u_d \end{pmatrix} \cdot \begin{array}{cl} \left(\begin{matrix} u_j \\ -u_i \end{matrix} \right) & \begin{matrix} \rightarrow \text{row } i \\ \rightarrow \text{row } j \end{matrix} \end{array} = \text{Const}$$

Equivalently, $$\frac{\dot{u}_i u_j - \dot{u}_j u_i}{(1-|u|^2)^2} = C_{ij} \qquad (6)$$

where C_{ij} is a constant. From (4) and (6), we get

$$|\dot{u}|^2 = C_1^2 (1-|u|^2)^2, \qquad (7)$$

$$\dot{u}_i u_j - \dot{u}_j u_i = C_{ij} (1-|u|^2)^2 \quad \forall 1 \le i < j \le d \qquad (8)$$

Then from (8) we get

$$\sum_{1\le i<j\le d} (\dot{u}_i u_j - \dot{u}_j u_i)^2 + \left(\sum_{i=1}^{d} \dot{u}_i u_i\right)^2 = \left(\sum_{1\le i<j\le d} C_{ij}^2\right)(1-|u|^2)^4 + \left(\frac{1}{2}\frac{d}{dt}(|u|^2)\right) \qquad (9)$$

Moreover, we have the identity (16) on page 11:

$$\sum_{1\le i<j\le d} (\dot{u}_i u_j - \dot{u}_j u_i)^2 + \left(\sum_{i=1}^{d} \dot{u}_i u_i\right)^2 = |\dot{u}|^2 |u|^2 \qquad (10)$$

Comparing (9) and (10) we get

$$\left(\sum_{1\le i<j\le d} C_{ij}^2\right)(1-|u|^2)^4 + \left(\frac{1}{2}\frac{d}{dt}(1-|u|^2)\right)^2 = |\dot{u}|^2|u|^2 \overset{(7)}{=\!=} C_1^2 (1-|u|^2)^2 |u|^2$$

Put $C = \sum_{1\le i<j\le d} C_{ij}^2 \geqslant 0$ and $g(t) = 1-|u(t)|^2$. We get

$$C g^4 + \left(\frac{1}{2}\frac{dg}{dt}\right)^2 = C_1^2 g^2 (1-g)$$

Then $$\left(\frac{1}{2}\frac{dg}{dt}\right)^2 = g^2 [C_1^2(1-g) - C g^2]$$

Thus, $$\frac{1}{2}\frac{dg}{dt} = \pm g \sqrt{C_1^2(1-g) - C g^2} \qquad (11)$$

Now we'll consider two cases.

- $\underline{C=0}$ Then $C_{ij}=0$ for all $i<j$. Then by (6) we get

$$\dot u_i u_j - \dot u_j u_i = 0 \qquad \forall\, 1\leq i<j\leq d$$

Thus the vectors $0, u_i, u_j$ are colinear in $C^2(\overset{\mathbb{R}}{[a,b]},\mathbb{R})$ by considering the derivative of $\frac{u_i}{u_j}$. Because this is true for all $1\leq i<j\leq n$, the vectors $0, u_1, \dots, u_d$ are colinear in $C^2(\overset{\mathbb{R}}{[a,b]},\mathbb{R})$. Thus, there exists a scalar function $f:\mathbb{R}\to\mathbb{R}$ such that $u=(u_1,\dots,u_d)=f(t)\vec{v}$, where $\vec{v}$ is a constant unit vector in $\mathbb{R}^d$. Note that $|f(t)|=|u|<1$. Thus f is a map from $\mathbb{R}$ to $(-1,1)$. Then the Euler-Lagrange equations (3) becomes

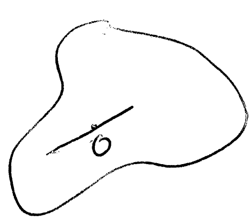
$$\frac{d}{dt}\left(\frac{\dot f}{(1-f^2)^2}\right) - \frac{2(\dot f)^2 f}{(1-f^2)^3} = 0.$$

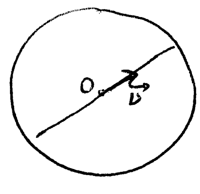
Now we return to the case $d=1$. We proved earlier at Eq. (4'') that

$$f(t) = \tanh(C_1 t + C_2) \quad \forall t\in\mathbb{R}.$$

Therefore, $\quad u(t) = \tanh(C_1 t + C_2)\,\vec{v} \qquad \forall t\in\mathbb{R}.$

Note that the curve $u=u(t)$ passes through the origin in $\mathbb{R}^d$.





- $\underline{C>0}$ Because $g(t)=1-|u(t)|^2$ and $u(t)\in B_d(0,1)$, we have

$$0<g(t)\leq 1 \quad \forall t\in\mathbb{R}.$$

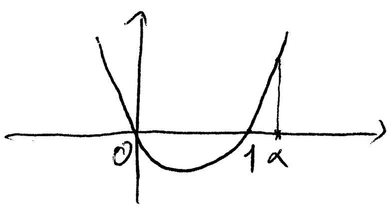
Note that $g(t) < 1$ because otherwise there exists $t_0 \in \mathbb{R}$ with $u(t_0) = 0$; then all ~~coef~~ constants C_{ij} are zero when we evaluate both sides of Eq. (6) at $t = t_0$. This contradicts the fact that $C > 0$.

If $g : \mathbb{R} \to (0,1)$ is a function satisfying (11) then

$$C_1^2(1-g) - Cg^2 \geq 0$$

Thus,

$$\left(\frac{1}{g}\right)^2 - \left(\frac{1}{g}\right) \geq \frac{C}{C_1^2} > 0 \qquad (12)$$



Let $\alpha > 1$ be the solution of $\alpha^2 - \alpha = \frac{C}{C_1^2}$. Since $0 < g(t) < 1$, we have $\frac{1}{g} > 1$. Thus the condition (12) is equivalent to $\frac{1}{g} \geq \alpha$. Thus,

$$0 < g(t) \leq \frac{1}{\alpha} < 1 \qquad \forall t \in \mathbb{R}.$$

From (11), we get

$$\int \frac{dg}{g\sqrt{C_1^2(1-g) - Cg^2}} = \int \pm 2\, dt$$

Hence,

$$\log\left(\frac{C_1^2(2-g) + 2C_1\sqrt{C_1^2(1-g) - Cg^2}}{g} \right) = \pm 2C_1 t + C_2 \qquad (13)$$

where C_2 is a constant. Because $0 < g(t) \leq \frac{1}{\alpha} < 1$, we have

$$\frac{C_1^2(2-g) + 2C_1\sqrt{C_1^2(1-g) - Cg^2}}{g} \geq \frac{C_1^2(2-g)}{g} \geq \frac{C_1^2}{g} \geq C_1^2\alpha > 0$$

Thus, LHS(13) $\geq \log(C_1^2\alpha) > -\infty$ for all $t \in \mathbb{R}$.

However, with a uniform choice of plus or minus sign on the right hand side of (13), RHS(13) ranges over all values in $\mathbb{R}$. This is a contradiction. In conclusion, all solution $u : \mathbb{R} \longrightarrow B_d(0,1)$ to the Euler-Lagrange equations (3) are

$$u(t) = \tanh(C_1 t + C_2)\,\vec{v},$$

where $\vec{v}$ is a constant unit vector in $\mathbb{R}^d$, and C_1, C_2 are constant numbers.