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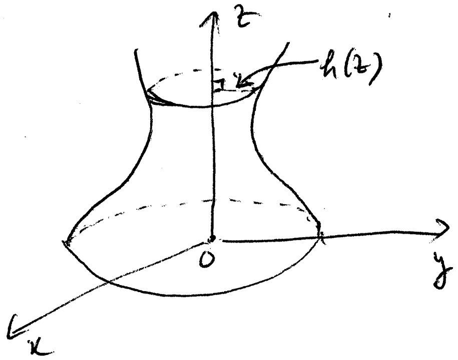
Math 8385: Calculus of Variations

Homework #3

(1) Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth positive function and denote

$$\Sigma = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2 = f(z)\}.$$

We'll first write down the equations of geodesics on Σ. Put $h(z):=\sqrt{f(z)}$, which is also smooth and positive. We write Σ as a union of open subsets



$$\Sigma = \Sigma_1 \cup \Sigma_2, \text{ where}$$

$$\Sigma_1 = \{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\neq(-h(z),0)\},$$

$$\Sigma_2 = \{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\neq(h(z),0)\}.$$

Define the maps

$$f_1:\mathbb{R}\times(-\pi,\pi)\to\Sigma_1,\quad f_1(z,\theta) = (h(z)\cos\theta,\ h(z)\sin\theta,\ z),$$

$$f_2:\mathbb{R}\times(0,2\pi)\to\Sigma_2,\quad f_2(z,\theta) = (h(z)\cos\theta,\ h(z)\sin\theta,\ z).$$

Then $\{(\Sigma_1,f_1),(\Sigma_2,f_2)\}$ forms an atlas on Σ. We now proceed to compute the metric tensor on Σ.

On Chart Σ_1

$$g_{11}(z,\theta) = \frac{\partial f_1}{\partial z}\cdot\frac{\partial f_1}{\partial z} = \left(\frac{\partial}{\partial z}(h(z)\cos\theta)\right)^2 + \left(\frac{\partial}{\partial z}(h(z)\sin\theta)\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2$$

$$= h'(z)^2+1,$$

$$g_{12}(z,\theta) = g_{21}(z,\theta) = \frac{\partial f_1}{\partial z}\cdot\frac{\partial f_1}{\partial \theta} = \frac{\partial}{\partial z}(h(z)\cos\theta)\frac{\partial}{\partial\theta}(h(z)\cos\theta) +$$

$$+\frac{\partial}{\partial z}(h(z)\sin\theta)\frac{\partial}{\partial\theta}(h(z)\sin\theta) + \frac{\partial z}{\partial z}\frac{\partial z}{\partial\theta}$$

$$= h'(z)(\cos\theta)h'(z)(-\sin\theta) + h'(z)(\sin\theta)h(z)(\cos\theta) = 0,$$

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$$g_{22}(z,\theta) = \frac{\partial f_1}{\partial \theta} \cdot \frac{\partial f_1}{\partial \theta} = \left(\frac{\partial}{\partial \theta}(h(z)\cos\theta)\right)^2 + \left(\frac{\partial}{\partial\theta}(h(z)\sin\theta)\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$
$$= \left(h(z)(-\sin\theta)\right)^2 + \left(h(z)\cos\theta\right)^2$$
$$= h(z)^2.$$

<u>On chart Σ_2</u> The expressions for g_{ij} are the same as those in chart Σ_1, except that the range of θ is now $(0, 2\pi)$ instead of $(-\pi, \pi)$. Namely,

$$(g_{ij}) = \begin{pmatrix} g_{11}(z,\theta) & g_{12}(z,\theta) \\ g_{21}(z,\theta) & g_{22}(z,\theta) \end{pmatrix} = \begin{pmatrix} h'(z)^2+1 & 0 \\ 0 & h(z)^2 \end{pmatrix} \qquad (1).$$

The inverse matrix of (g_{ij}) is

$$(g^{ij}) = \begin{pmatrix} g^{11}(z,\theta) & g^{12}(z,\theta) \\ g^{21}(z,\theta) & g^{22}(z,\theta) \end{pmatrix} = \begin{pmatrix} \frac{1}{h'(z)^2+1} & 0 \\ 0 & \frac{1}{h(z)^2} \end{pmatrix} \qquad (2)$$

Next, we compute the Christoffel symbols. To be able to use Eistein's summation convention, we'll use the name (z_1, z_2) interchangeably with (z,θ).

By definition, $\Gamma_{ij}^{l} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{ik}}{\partial z_j} - \frac{\partial g_{ij}}{\partial z_k} + \frac{\partial g_{jk}}{\partial z_i}\right) \quad \forall 1 \le i,j,l \le 2.$

Since $g^{kl} = 0$ if $k \ne l$, we have

$$\Gamma_{ij}^{l} = \frac{1}{2} g^{ll}\left(\frac{\partial g_{il}}{\partial z_j} - \frac{\partial g_{ij}}{\partial z_l} + \frac{\partial g_{jl}}{\partial z_i}\right) \qquad (3)$$

Using (3), we get $\Gamma_{11}^{1} = \frac{1}{2} g^{11}\left(\frac{\partial g_{11}}{\partial z_1} - \frac{\partial g_{11}}{\partial z_1} + \frac{\partial g_{11}}{\partial z_1}\right) = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial z}$

$$= \frac{1}{2}\frac{1}{h'(z)^2+1}\frac{d}{dz}[h'(z)^2+1]$$
$$= \frac{h'(z)h''(z)}{h'(z)^2+1}$$

$$\Gamma_{11}^{2} = \frac{1}{2} g^{22}\Big(\underbrace{\frac{\partial g_{12}}{\partial z_1}}_{=0} - \underbrace{\frac{\partial g_{11}}{\partial z_2}}_{=0} + \underbrace{\frac{\partial g_{12}}{\partial z_1}}_{=0}\Big) = 0,$$

$$\Gamma_{12}^{1} = \frac{1}{2} g^{11}\Big(\underbrace{\frac{\partial g_{11}}{\partial z_2}}_{=0} - \underbrace{\frac{\partial g_{12}}{\partial z_1}}_{=0} + \underbrace{\frac{\partial g_{21}}{\partial z_1}}_{=0}\Big) = 0,$$

$$\Gamma_{12}^{2} = \frac{1}{2} g^{22}\Big(\underbrace{\frac{\partial g_{12}}{\partial z_2}}_{=0} - \underbrace{\frac{\partial g_{12}}{\partial z_2}}_{=0} + \frac{\partial g_{22}}{\partial z_1}\Big) = \frac{1}{2} g^{22}\frac{\partial g_{22}}{\partial z}$$

$$= \frac{1}{2}\frac{1}{h(z)^2}\frac{d}{dz}\left[h(z)^2\right] = \frac{h'(z)}{h(z)}.$$

$$\Gamma_{22}^{1} = \frac{1}{2} g^{11}\Big(\underbrace{\frac{\partial g_{21}}{\partial z_2}}_{=0} - \frac{\partial g_{22}}{\partial z_1} + \underbrace{\frac{\partial g_{21}}{\partial z_2}}_{=0}\Big) = -\frac{1}{2} g^{11}\frac{\partial g_{22}}{\partial z}$$

$$= -\frac{1}{2}\frac{1}{h'(z)^2+1}\frac{d}{dz}\left[h(z)^2\right] = -\frac{h'(z)h(z)}{h'(z)^2+1}.$$

$$\Gamma_{22}^{2} = \frac{1}{2} g^{22}\Big(\underbrace{\frac{\partial g_{22}}{\partial z_2}}_{=0} - \underbrace{\frac{\partial g_{22}}{\partial z_2}}_{=0} + \underbrace{\frac{\partial g_{22}}{\partial z_2}}_{=0}\Big) = 0.$$

In conclusion,

$$\begin{cases} \Gamma_{11}^{1}(z,\theta) = \dfrac{h''(z)\,h'(z)}{h'(z)^2+1}, \\[2ex] \Gamma_{12}^{2}(z,\theta) = \Gamma_{21}^{2}(z,\theta) = \dfrac{h'(z)}{h(z)}, \\[2ex] \Gamma_{22}^{1}(z,\theta) = -\dfrac{h'(z)h(z)}{h'(z)^2+1}, \\[2ex] \Gamma_{ij}^{\ell}(z,\theta) = 0 \quad \text{for other tuples } (i,j,\ell). \end{cases} \tag{4}$$

Let $\gamma(t) = (\gamma_1(t), \gamma_2(t)) = (z(t), \theta(t))$ be a geodesic on Σ lying entirely in Σ_1 or Σ_2.

Then by Lemma 2.1.2, page 39, Jost-Li Jost, $\gamma(t)$ satisfies the equations

$$\ddot{\gamma}_i(t) + \Gamma^i_{jk}(\gamma(t))\,\dot{\gamma}_j(t)\,\dot{\gamma}_k(t) = 0,$$

which can be expanded as

$$\ddot{z}(t) + \Gamma^1_{jk}(z(t),\theta(t))\,\dot{\gamma}_j(t)\,\dot{\gamma}_k(t) = 0,$$

$$\ddot{\theta}(t) + \Gamma^2_{jk}(z(t),\theta(t))\,\dot{\gamma}_j(t)\,\dot{\gamma}_k(t) = 0.$$

Thanks to (4), the above equations become

$$\ddot{z}(t) + \Gamma^1_{11}(z(t),\theta(t))\,\dot{z}(t)\,\dot{z}(t) + \Gamma^1_{22}(z(t),\theta(t))\,\dot{\theta}(t)\,\dot{\theta}(t) = 0,$$

$$\ddot{\theta}(t) + \Gamma^2_{12}(z(t),\theta(t))\,\dot{z}(t)\dot{\theta}(t) + \Gamma^2_{21}(z(t),\theta(t))\,\dot{z}(t)\,\dot{\theta}(t) = 0.$$

Equivalently,

$$\ddot{z}(t) + \frac{h''(z(t))\,h'(z(t))}{h'(z(t))^2+1}(\dot{z}(t))^2 - \frac{h'(z(t))\,h(z(t))}{h'(z(t))^2+1}(\dot{\theta}(t))^2 = 0,$$

$$\ddot{\theta}(t) - 2\frac{h'(z(t))}{h(z(t))}\,\dot{z}(t)\,\dot{\theta}(t) = 0$$

In a more compact form, these equations are

$$\begin{cases} \ddot{z} + \dfrac{h''(z)h'(z)}{h'(z)^2+1}(\dot{z})^2 - \dfrac{h'(z)h(z)}{h'(z)^2+1}(\dot{\theta})^2 = 0, & (5) \\[2ex] \ddot{\theta} - \dfrac{2h'(z)}{h(z)}\dot{z}\dot{\theta} = 0. & (6) \end{cases}$$

A continuous curve $\frac{x}{y} = \text{const}$ on Σ corresponds to a line $\theta = \text{const}$ in either local chart Σ_1 or Σ_2. We'll show that ~~for~~ any two points $(z,\theta) = (a,\theta_0)$ and $(z,\theta) = (b,\theta_0)$, both in Σ_1 or both in Σ_2, can be connected by a geodesic $\gamma(t) = (z(t),\theta_0)$ with $z(0) = a$, $z(1) = b$.

For any curve $\gamma(t) = (z(t), \theta_0)$, equation (6) is automatically satisfied because $\dot{\theta} \equiv 0$. Equation (5) now becomes

$$\ddot{z} + \frac{h''(z)\,h'(z)}{h'(z)^2+1}(\dot{z})^2 = 0$$

$$\Leftrightarrow \ddot{z}\,(h'(z)^2+1) + h''(z)\,h'(z)(\dot{z})^2 = 0$$

$$\Leftrightarrow \frac{d}{dt}\left[\dot{z}\,(h'(z)^2+1)\right] = 0$$

$$\Leftrightarrow \dot{z}\,(h'(z)^2+1) \equiv C_1 \quad (\text{constant})$$

Multiplying both sides by dt and taking integral both sides, we get

$$\int_0^{z(t)} (h'(s)^2+1)\,ds = C_1 t + C_2 \qquad (7)$$

Define the map $H : \mathbb{R} \to \mathbb{R}$, $H(z) = \int_0^z (h'(s)^2+1)\,ds$. Then H is smooth and increasing in $\mathbb{R}$. Moreover, $H(z) \geq z$ if $z \geq 0$ and

$$H(z) = -\int_z^0 (h'(s)^2+1)\,ds \leq -\int_z^0 ds = -z \qquad \text{if } z < 0.$$

Thus $\lim_{z\to\infty} H(z) = \infty$ and $\lim_{z\to-\infty} H(z) = -\infty$. Thus H is surjective. Thus H is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Put $\tilde{H} = H^{-1}$. Then $\tilde{H}$ is also differentiable and

$$\tilde{H}'(z) = \frac{1}{H'(\tilde{H}(z))} = \frac{1}{h'(\tilde{H}(z))^2+1} \qquad (8)$$

Since $RHS(8) \in C^1(\mathbb{R})$, $\tilde{H}' \in C^1(\mathbb{R})$. Thus $\tilde{H} \in C^2(\mathbb{R})$. Then $RHS(8) \in C^2(\mathbb{R})$. Thus, $\tilde{H}' \in C^2(\mathbb{R})$. Then $\tilde{H} \in C^3(\mathbb{R})$.

From (7), we get $H(z(t)) = C_1 t + C_2$. This is equivalent to

$$z(t) = \tilde{H}(C_1 t + C_2) \qquad (9)$$

which is in $C^3(\mathbb{R})$. For any choices of constants C_1 and C_2, $z(t)$ given by (9) is guaranteed to satisfy equation (5). For any two points (a, θ_0) and (b, θ_0) ~~both~~ lying entirely in Σ_1 or Σ_2, we choose $C_2 = H(a)$, $C_1 = H(b) - H(a)$. Then $z(t)$ given by (9) satisfies $z(0) = a$ and $z(1) = b$. Therefore, the curve $\gamma(t) = (z(t), \theta_0)$ is a geodesic on Σ connecting (a, θ_0) and (b, θ_0).

Next, we'll answer the question when a curve $z \equiv$ const will be a geodesic. More sepecifically, taking any $z_0 \in \mathbb{R}$, $\theta_1, \theta_2 \in (-\pi, \pi)$ (or $\theta_1, \theta_2 \in (0, 2\pi)$), $\theta_1 \neq \theta_2$, we'll answer the question when there will be a geodesic $\gamma(t) = (z_0, \theta(t))$ with $\theta(0) = \theta_1$ and $\theta(1) = \theta_2$. In such a case, $\dot{z} \equiv 0$. Then the equations (5) and (6) become

$$\begin{cases} h'(z_0)\, h(z_0)\, (\dot{\theta}(t))^2 \equiv 0 & (10) \\ \ddot{\theta}(t) \equiv 0 & (11) \end{cases}$$

Equation (11) implies $\theta(t) = (1-t)\theta_1 + t\theta_2$. Thus, $\dot{\theta}(t) \equiv \theta_2 - \theta_1 \neq 0$. Also, $h(z_0) > 0$ since h is a positive function. Then (10) is equivalent to $h'(z_0)$. Since $h(z) = \sqrt{f(z)}$, we have $h'(z) = \dfrac{f'(z)}{2\sqrt{f(z)}}$. Then that $h'(z_0) = 0$ is equivalent to that $f'(z_0) = 0$.

We conclude that the curve $\gamma(t) = (z_0, (1-t)\theta_1 + t\theta_2)$ is a geodesic on Σ if and only if $f'(z_0) = 0$.

② Let M be a C^3-submanifold of $\mathbb{R}^d$ that is diffeomorphic to S^2, say via a diffeomorphism $h: S^2 \to M$. We'll show that for any point $p \in M$, there is a non-constant geodesic on M which starts and ends at p.

We restate Theorem 2.3.1, Jost-Li Jost, page 51, as follows.

<u>Lemma 1</u> If M is a compact C^3-submanifold of $\mathbb{R}^d$ then there exists a number $\varepsilon_0 > 0$ such that ~~for~~ any two points $p_1, p_2 \in M$ with $d(p_1, p_2) \leq \varepsilon_0$ can be connected by a unique shortest geodesic arc, which is thus of length $d(p_1, p_2)$. Moreover, this geodesic depends continuously on two of its end points.

Note that $d(p_1, p_2)$ denotes the distance between p_1 and p_2 on M, namely

$$d(p_1, p_2) := \inf \{ L(c) \mid c : [0,1] \to M, \text{ continuous}, c(0) = p_1, c(1) = p_2 \},$$

where $L(c)$ is the length of path c, i.e. $L(c) = \int_0^1 |\dot{c}(t)| dt$.

We can elaborate the meaning of the last statement in the above lemma as follows. Suppose that there are two sequences (p_n) and (q_n) in M such that

- $p_n \to p_0 \in M$, $q_n \to q_0 \in M$
- $d(p_n, q_n) \leq \varepsilon_0 \quad \forall n \in \mathbb{N}$

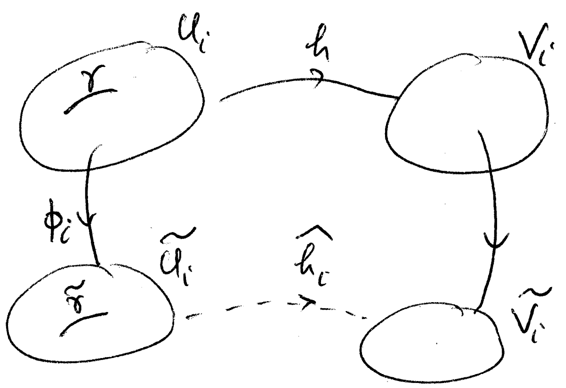
Let $\gamma_n : [0,1] \to M$ be the unique shortest geodesic on M from p_n to q_n.
Let $\gamma_0 : [0,1] \to M$ be the unique ~~geo~~ shortest geodesic on M from p_0 to q_0.
Then $\sup\limits_{t \in [0,1]} |\gamma_n(t) - \gamma_0(t)| \to 0$ as $n \to \infty$.

Return to the problem. We'll prove the following lemma, which is the inequality (2.3.2), Jost-Li Jost, page 54, but were not proved therein.

[Lemma 2 Let M and $h: S^2 \to M$ be given as at the beginning of the problem. Then there exists a constant $K_0 > 0$ such that for any curve $\gamma: [0,1] \to S^2$, we have $L(h \circ \gamma) \le K_0 L(\gamma)$.]

Proof of lemma 2

Note that $h \circ \gamma$ is a curve on M, while γ is a curve on S^2. Suppose that S^2 has C^1 atlas $(U_i, \varphi_i: U_i \to \tilde{U}_i)$, $1 \le i \le m$, where $\tilde{U}_i$ is an open subset of $\mathbb{R}^2$; and that M has C^1-atlas $(V_i; \psi_i: V_i \to \tilde{V}_i)$, $1 \le i \le m$, where $\tilde{V}_i$ is an open subset of $\mathbb{R}^2$. Suppose also that $h(U_i) \subset V_i$.



Then we get a local representation of h on U_i, namely $\hat{h}_i : \tilde{U}_i \to \tilde{V}_i$,

$$\hat{h}_i = \psi_i \circ h \circ \varphi_i^{-1}.$$

We have $\hat{h}_i \in C^1$, $\psi_i \in C^1$, $\phi_i \in C^1$.

By shrinking $U_i, V_i, \tilde{U}_i, \tilde{V}_i$ if necessary, we can assume that

- $|D\hat{h}_i| \le K_1$ on $\tilde{U}_i$, $\forall 1 \le i \le m$,
- $|D(\psi_i^{-1})| \le K_2$ on $\tilde{V}_i$, $\forall 1 \le i \le m$,
- $|D\phi_i| \le K_3$ on U_i, $\forall 1 \le i \le m$.

Put $K_0 = K_1 K_2 K_3$. Let γ be any ~~path~~ differentiable path on S^2, which lies entirely in some chart V_i. Put $\tilde{\gamma} = \phi_i \circ \gamma$. Then by the chain rule, $\dot{\tilde{\gamma}}(t) = (D\phi)(\gamma(t)) \cdot \dot{\gamma}(t)$. Thus, $|\dot{\tilde{\gamma}}(t)| \le |(D\phi)(\gamma(t))| \, |\dot{\gamma}(t)| \le K_3 |\dot{\gamma}(t)|$.

Then $L(\tilde{\gamma}) = \int_0^1 |\dot{\tilde{\gamma}}(t)| dt \le K_3 \int_0^1 |\dot{\gamma}(t)| dt = K_3 L(\gamma)$.

Similarly, $\frac{d}{dt}(\hat{h}_i\circ\tilde{\gamma}) = D\hat{h}_i(\tilde{\gamma}(t))\cdot\dot{\tilde{\gamma}}(t)$. Thus,

$$\left|\frac{d}{dt}(\hat{h}_i\circ\tilde{\gamma})\right| \leq |D\hat{h}_i(\tilde{\gamma}(t))|\,|\dot{\tilde{\gamma}}(t)| \leq K_1|\dot{\tilde{\gamma}}(t)| \leq K_1K_3|\dot{\gamma}(t)|.$$

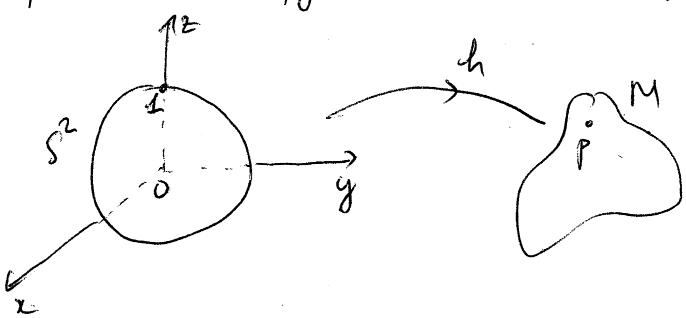
Hence,
$$L(\hat{h}_i\circ\tilde{\gamma}) \leq \int_0^1 \left|\frac{d}{dt}(\hat{h}_i\circ\tilde{\gamma})\right| dt \leq K_1K_3\int_0^1|\dot{\gamma}(t)|\,dt = K_1K_3L(\gamma).$$

Put $\bar{\gamma} = \hat{h}_i\circ\tilde{\gamma}$. Because $h = \psi_i^{-1}\circ\hat{h}_i\circ\varphi_i$, we have $h\circ\gamma = \psi_i^{-1}\circ\bar{\gamma}$. Then by the chain rule, $\frac{d}{dt}(h\circ\gamma) = \frac{d}{dt}(\psi_i^{-1}\circ\bar{\gamma}) = (D\psi_i^{-1})(\bar{\gamma}(t))\cdot\dot{\bar{\gamma}}(t)$.

Then
$$\left|\frac{d}{dt}(h\circ\gamma)\right| \leq |(D\psi_i^{-1})(\bar{\gamma}(t))|\,|\dot{\bar{\gamma}}(t)| \leq K_2|\dot{\bar{\gamma}}(t)| \leq \underbrace{K_1K_2K_3}_{=K_0}|\dot{\gamma}(t)|.$$

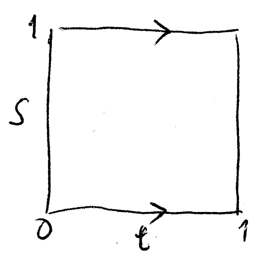
Thus,
$$L(h\circ\gamma) = \int_0^1\left|\frac{d}{dt}(h\circ\gamma)\right| dt \leq K_0\int_0^1|\dot{\gamma}(t)|\,dt = K_0\,L(\gamma).$$
∎

Note: In the proof of the previous lemma, we didn't use the ~~fa~~ hypothesis $h^{-1}\in C^1$. Return to the problem. Take any point $p\in M$. We'll show that there is a nontrivial geodesic on M which starts and ends at p. We identify S^2 with $\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$.



By composing h with a rotation in $\mathbb{R}^3$, centered at O, we can assume that $h(1,0,0) = p$.

Denote $\gamma(s,t) := h(\cos(2\pi t), \sin(2\pi t)\cos(2\pi s), \sin(2\pi t)\sin(2\pi s))$ for $s,t\in[0,1]$.



Consider the square $[0,1)\times[0,1)$ with edge-identification

$(s,0)\sim(0,0)$ $\quad\forall s\in[0,1)$,

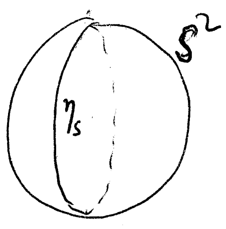
$(s,1)\sim(0,1)$ $\quad\forall s\in[0,1)$,

$(0,t)\sim(1,t)$ $\quad\forall t\in[0,1)$.

Then $[0,1]\times[0,1]/\sim$ is homeomorphic to S^2. The definition of γ allows us to view it as a function from $[0,1]\times[0,1]/\sim$ to M, or even as a function from S^2 to M.

We'll write $\gamma_s(t) := \gamma(s,t)$. Then γ_s is a curve on M which starts and ends at p. Put

$$\eta_s(t) = \big(\cos(2\pi t),\ \sin(2\pi t)\cos(2\pi s),\ \sin(2\pi t)\sin(2\pi s)\big) \quad \forall s,t\in[0,1].$$

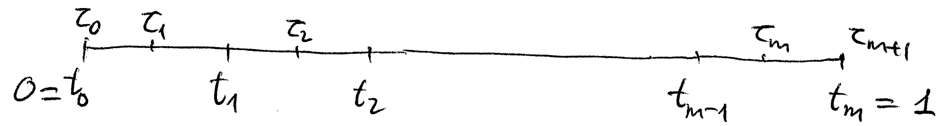


Then η_s is a great circle on S^2 passing through $(1,0,0)$ and $(-1,0,0)$. Then by the definition of γ_s, we have

$$\gamma_s = h\circ\eta_s.$$

Let $\varepsilon_0 > 0$ be the constant number mentioned in Lemma 1 above. Take any $m\in\mathbb{N}$, $m > K_0\pi\varepsilon_0^{-1}$, where K_0 is the constant mentioned in Lemma 2. Put $t_i = \frac{i}{m}$ for every $0\le i\le m$,

$$\tau_0 = 0,\quad \tau_{m+1} = 1,\quad \tau_i = \frac{2i-1}{2m} \quad \text{for every } 1\le i\le m.$$



We have
$$\begin{aligned} d(\gamma_s(t_i),\gamma_s(t_{i+1})) &= d(h\circ\eta_s(t_i),\, h\circ\eta_s(t_{i+1})) \\ &\le L\big(h\circ\eta_s|_{[t_i,t_{i+1}]}\big) \\ &\le K_0 L\big(\eta_s|_{[t_i,t_{i+1}]}\big) \quad \text{(by Lemma 2).} \end{aligned}$$

Because η_s is a great circle on S^2, $L\big(\eta_s|_{[t_i,t_{i+1}]}\big) = \pi(t_{i+1}-t_i)$.

Then $d(\gamma_s(t_i), \gamma_s(t_{i+1})) \leq K_0\pi(t_{i+1} - t_i) \leq \frac{K_0\pi}{m} < \varepsilon_0$.

Also, $d(\gamma_s(\tau_i), \gamma_s(\tau_{i+1})) = d(h\circ\eta_s(\tau_i), h\circ\eta_s(\tau_{i+1}))$

$$\leq L(h\circ\eta_s|_{[\tau_i,\tau_{i+1}]})$$

$$\leq K_0\, L(\eta_s|_{[\tau_i,\tau_{i+1}]}) \quad \text{(by Lemma 2)}$$

$$= K_0\pi(\tau_{i+1} - \tau_i)$$

$$\leq K_0\pi\frac{1}{m} < \varepsilon_0.$$

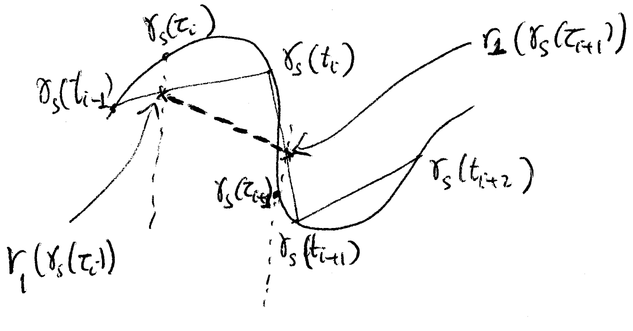
Therefore, we can connect $\gamma_s(t_i)$ to $\gamma_s(t_{i+1})$ by a unique shortest geodesic on M. Also, we can connect $\gamma_s(\tau_i)$ to $\gamma_s(\tau_{i+1})$ by a unique shortest geodesic on M. For any curve $\lambda : [0,1] \to M$,

- If $d(\lambda(t_i), \lambda(t_{i+1})) \leq \varepsilon_0 \ \forall i$ then we define a reductive operator of type 1, namely r_1, as $r_1(\lambda)|_{[t_i, t_{i+1}]} =$ the unique shortest geodesic from $\lambda(t_i)$ to $\lambda(t_{i+1})$.

- If $d(\lambda(\tau_i), \lambda(\tau_{i+1})) \leq \varepsilon_0 \ \forall i$ then we define a reductive operator of type 2, namely r_2, as $r_2(\lambda)|_{[\tau_i,\tau_{i+1}]} =$ the unique shortest geodesic from $\lambda(\tau_i)$ to $\lambda(\tau_{i+1})$.

With these definitions, we can define $r_1(\gamma_s)$. It is <u>not</u> clear why

$$d(r_1(\gamma_s(\tau_i)), r_1(\gamma_s(\tau_{i+1}))) \leq \varepsilon_0 \ \forall i \qquad (*)$$



In Jost–Li Jost, page 55 (in the statement of Lemma 2.3.2), they assumed that (*) holds so that they can define $r_2(r_1(\gamma_s$

This is a nontrivial assumption even if γ_s is parametrized by arclength. Denote $r(\gamma_s) := r_2(r_1(\gamma_s))$. Jost - Li Jost assumed that the pair of points ~~t=t~~ at $t=t_i$ and $t=t_{i+1}$ remains to be of distance $\leq \varepsilon_0$ after each reduction step. The authors made the same assumption for the pair of points at $t=\tau_i$ and $t=\tau_{i+1}$. Under those assumptions, we can define

$$r^n(\gamma_s) := \underbrace{r \circ r \circ \dots \circ r}_{n \text{ times}}(\gamma_s)$$

In the remaining of the proof, we'll use the following lemmas.

<u>Lemma 3</u> Let $\gamma : [0,1] \to M$ be a curve such that $d(\gamma(t_i), \gamma(t_{i+1})) \leq \varepsilon_0$ and $d(\gamma(\tau_i), \gamma(\tau_{i+1})) \leq \varepsilon_0$ $\forall i$. Then $L(r(\gamma)) \leq L(\gamma)$. The equality holds if and only if γ is a geodesic.

Note: this is Lemma 2.3.1, page 55, Jost - Li Jost.

<u>Lemma 4</u> For each $s \in [0,1]$, the sequence $\{r^n(\gamma_s)\}_{n \in \mathbb{N}}$ has a subsequence which converges uniformly to a geodesic which starts and ends at p.

Note: this is Lemma 2.3.2, page 55, Jost - Li Jost.

<u>Lemma 5</u> (Dini's Theorem)

Suppose that a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ satisfies

- $\forall s \in [0,1]$, $\lim_{n \to \infty} f_n(s) = f(s) \in \mathbb{R}$,
- $\forall s \in [0,1]$, $\forall n \in \mathbb{N}$, $f_n(s) \geq f_{n+1}(s)$.
- f is continuous on $[0,1]$.

Then $f_n \to f$ uniformly on $[0,1]$ as $n \to \infty$.

<u>Proof of Lemma 5</u>

Put $g_n(s) = f_n(s) - f(s) \geq 0$. Then $g_n(s) \geq g_{n+1}(s)$ and $g_n(s) \to 0$ $\forall s \in [0,1]$.
Moreover, g_n is continuous since f_n and f are continuous. Take any $\varepsilon > 0$.
Put $K_n = \{s \in [0,1] : g_n(s) \geq \varepsilon\} = g_n^{-1}([\varepsilon, \infty))$. Then K_n is closed in $[0,1]$.
Since $g_n(s) \geq g_{n+1}(s)$, we have $K_1 \supset K_2 \supset \cdots \supset K_n \supset K_{n+1} \supset \cdots$

For each $s \in [0,1]$, $\lim_{n\to\infty} g_n(s) = 0$. Thus $s \notin K_n$ for all $n > N_\varepsilon$. Thus $s \notin \bigcap_{n=1}^{\infty} K_n$. This means $\bigcap_{n=1}^{\infty} K_n = \phi$. This is an intersection of closed subsets in a compact set $[0,1]$. Thus there exist $K_{i_1}, \dots, K_{i_\ell}$ with $i_1 < \dots < i_\ell$ such that $\bigcap_{n=1}^{\infty} K_n = K_{i_1} \cap \dots \cap K_{i_\ell} = K_{i_\ell}$. Thus $K_{i_\ell} = 0$. Then $K_n = 0$ $\forall n \geq i_\ell$.
Thus, for each $n > i_\ell$, $g_n(s) < \varepsilon$ for all $s \in [0,1]$. This means $g_n \to 0$ uniformly in $s \in [0,1]$ as $n \to \infty$. Thus, $f_n \to f$ uniformly on $[0,1]$ as $n \to \infty$.

Return to the problem. We put $\kappa_1 = \sup_{0 \leq s \leq 1} \left(\lim_{n\to\infty} L(r^n(\gamma_s)) \right)$. Note that the limit exists because $L(r^0(\gamma_s)) = L(\gamma_s) \geq L(r(\gamma_s)) \geq \cdots \geq L(r^n(\gamma_s)) \geq \cdots \geq 0$.
Moreover, $\kappa_1 < \infty$ because

$$L(\gamma_s) = L(h \circ \eta_s) \leq K_0 L(\eta_s) \quad \text{(by Lemma 2)}$$
$$= K_0 2\pi < \infty \quad \forall s \in [0,1].$$

We put $f_n(s) = L(r^n(\gamma_s))$. Then f_n is continuous in $s \in [0,1]$ because γ_s is continuous on $s \in [0,1]$. Put $f(s) = \lim_{n\to\infty} L(r^n(\gamma_s))$. We <u>assume</u> that f is continuous in $[0,1]$. Jost-Li Jost didn't prove the continuity of f, while it is essential ~~to~~ in order to get the first inequality on page 58.
Although it is not clear why f should be continuous in $s \in [0,1]$, we will

make this assumption to continue the proof. By Lemma 3, we have $f_n(s) \geq f_{n+1}(s)$. Therefore, by Lemma 5, $f_n \to f$ uniformly in $s \in [0,1]$. Moreover, because $\kappa_1 = \sup_{0 \leq s \leq 1} f(s)$ and f is continuous there exists $s_0 \in [0,1]$ such that $\kappa_1 = f(s_0) = \lim_{n\to\infty} L(r^n(\gamma_{s_0}))$.

By Lemma 4, there exists an increasing sequence (n_k) in $\mathbb{N}$ so that $\{r^{n_k}(\gamma_{s_0})\}$ converges uniformly to a geodesic c which starts and ends at p. We have

$$L(c) = \lim_{k\to\infty} L(r^{n_k}(\gamma_{s_0})) = \kappa_1.$$

If $\kappa_1 > 0$ then $L(c) \neq 0$. Then c is a nonconstant curve, and we're done. We'll show that the case $\kappa_1 = 0$ is impossible. Suppose by contradiction that $\kappa_1 = 0$. Recall the notations

$$f_n(s) = L(r^n(\gamma_s)),$$

$$f(s) = \lim_{n\to\infty} L(r^n(\gamma_s)),$$

$$\kappa_1 = \sup_{0 \leq s \leq 1} f(s).$$

Because $\kappa_1 = 0$, $f(s) = 0$ for all $0 \leq s \leq 1$. We proved earlier by using Lemma 5 that f_n converges to $f = 0$ uniformly in $s \in [0,1]$. Thus,

$$L(r^n(\gamma_s)) \to 0 \text{ uniformly in } s \in [0,1] \text{ as } n \to \infty. \qquad (**)$$

Next, we'll show that for every $n \in \mathbb{N}$ and $q \in M$, there exist $s, t \in [0,1]$ such that $r^n(\gamma_s(t)) = q$.

Denote $r^n(\gamma)(s,t) := r^n(\gamma_s(t))$. Then $r^n(\gamma)$ is a function from $[0,1]\times[0,1]/\sim$ to M. We'll show that for every $n\in\mathbb{N}$, $r^n(\gamma)$ is surjective from $[0,1]\times[0,1]/\sim\ (\cong S^2)$ to M.

$$\boxed{\begin{array}{l}\underline{\text{Lemma 6}}\quad \text{Let } c:[0,1]\to M \text{ be a geodesic. Suppose that } a,b\in\mathbb{R} \\ \text{satisfy } at+b\in[0,1] \text{ for all } t\in[0,1]. \text{ Then the curve } \tilde{c}(t)=c(at+b) \\ \text{is also a geodesic on } M.\end{array}}$$

Proof of Lemma 6 Let γ be a coordinate representation of c. Then γ satisfies the geodesic equation $\ddot{\gamma}_i(t)+\Gamma^i_{jk}(\gamma(t))\,\dot{\gamma}_j(t)\,\dot{\gamma}_k(t)=0$.

The curve $\tilde{\gamma}(t)=\gamma(at+b)$ is the coordinate representation of $\tilde{c}$. We have

$$\ddot{\tilde{\gamma}}_i(t) = a^2\,\ddot{\gamma}_i(at+b),$$
$$\dot{\tilde{\gamma}}_j(t) = a\,\dot{\gamma}_j(at+b),$$
$$\dot{\tilde{\gamma}}_k(t) = a\,\dot{\gamma}_k(at+b).$$

Then $$\ddot{\tilde{\gamma}}_i(t)+\Gamma^i_{jk}(\tilde{\gamma}(t))\,\dot{\tilde{\gamma}}_j\,\dot{\tilde{\gamma}}_k = a^2\Big(\underbrace{\ddot{\gamma}_i(at+b)+\Gamma^i_{jk}(\gamma(at+b))\,\dot{\gamma}_j(at+b)\,\dot{\gamma}_k(at+b)}_{=0}\Big)$$
$$=0$$

Therefore, $\tilde{\gamma}$ is also a geodesic on M ∎

Return to the problem. For each $n\in\mathbb{N}$, we define a map $H_n : M\times[0,1]\to M$ as follows. Recall that we have the bijection

$(s,t)\in([0,1]\times[0,1])/\sim\ \to \gamma_s(t)\in M$.

$H_n(\cdot, 0) = id_M$. (and thus $H_n(\gamma_s(t), 0) = \gamma_s(t)$)

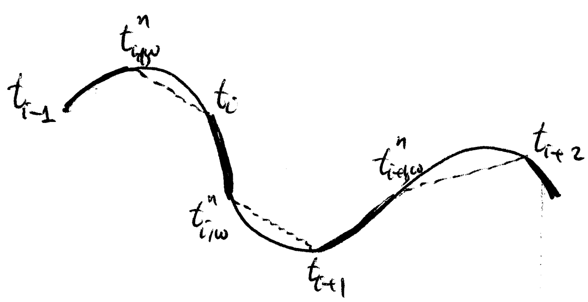
Put $t^n_{i,w} := t_i (2nw) + t_{i+1}(1-2nw)$,

$\tau^n_{i,w} := \tau_i (2nw) + \tau_{i+1}(1-2nw)$.

For any integer $0 \le k \le n-1$, we define for each $0 \le w \le \frac{1}{2n}$:

- $$H_n\left(\gamma_s(t), w+\frac{k}{n}\right) = \begin{cases} H_n\left(\gamma_s(t), \frac{k}{n}\right) & \text{if } t \in [t_i, t^n_{i,w}], \\ \text{geodesic from } H_n\left(\gamma_s(t^n_{i,w}), \frac{k}{n}\right) \text{ to } H_n\left(\gamma_s(t_{i+1}), \frac{k}{n}\right) & \\ \quad \text{which is parametrized by } t \in \cancel{[t_i, t^n_{i,w}]}\,[t^n_{i,w}, t_{i+1}]. & \end{cases}$$

This is the deformation from $r^n(\gamma_s)$ to $r_1 r^n(\gamma_s)$.

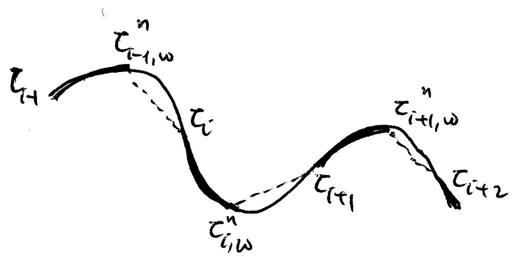


— $H_n\left(\gamma_s(t), w+\frac{k}{n}\right)$

— $H_n\left(\gamma_s(t), \frac{k}{n}\right)$

$$H_n\left(\gamma_s(t), w+\frac{k}{n}+\frac{1}{2n}\right) = \begin{cases} H_n\left(\gamma_s(t), \frac{k}{n}+\frac{1}{2n}\right) & \text{if } t \in [\tau_i, \tau^n_{i,w}], \\ \text{geodesic from } H_n\left(\gamma_s(\tau^n_{i,w}), \frac{k}{n}+\frac{1}{2n}\right) \text{ to} & \\ \quad H_n\left(\gamma_s(\tau_{i+1}), \frac{k}{n}+\frac{1}{2n}\right) \text{ which is} & \\ \quad \text{parametrized by } t \in [\tau^n_{i,w}, \tau_{i+1}]. & \end{cases}$$

This is the deformation from $r_1 r^n(\gamma_s)$ to $r_2 r_1 r^n(\gamma_s) = r^{n+1}(\gamma_s)$.



— $H_n\left(\gamma_s(t), w+\frac{k}{n}+\frac{1}{2n}\right)$

— $H_n\left(\gamma_s(t), \frac{k}{n}+\frac{1}{2n}\right)$

From this definition, we have

$$H_n\left(\gamma_s(t), \frac{k}{n} + \frac{1}{2n}\right) = r_1\left(H_n\left(\gamma_s(t), \frac{k}{n}\right)\right),$$

$$H_n\left(\gamma_s(t), \frac{k+1}{n}\right) = r_2\left(H_n\left(\gamma_s(t), \frac{k}{n} + \frac{1}{2n}\right)\right).$$

Consequently, $H_n(\gamma_s(t), 1) = r\left(H_n\left(\gamma_s(t), \frac{n-1}{n}\right)\right) = r^2\left(H_n\left(\gamma_s(t), \frac{n-2}{n}\right)\right)$

$$= \cdots = r^n\left(H_n(\gamma_s(t), 0)\right) = r^n(\gamma_s(t)) = (r^n \gamma$$

Therefore, H_n is a continuous homotopy from id_M to $r^n \circ \gamma$.

$$S^2 \times [0,1] \xrightarrow{h \times id} M \times [0,1] \xrightarrow{H_n} M \xrightarrow{h^{-1}} S^2$$

Then we get a homotopy from Id_{S^2} to $g : S^2 \to S^2$ where

$$g(x) = h^{-1} \circ H_n(\cdot, 1) \circ (h \times id)(x, 1)$$
$$= h^{-1} \circ H(h(x), 1).$$

Thus, $h \circ g(x) = H(h(x), 1)$. Now substitute x by $\eta_s(t)$ an

recall that $\gamma_s(t) = h(\eta_s(t))$. We get

$$(h \circ g)(\eta_s(t)) = H(\gamma_s(t), 1) = (r^n \gamma)(s,t) \quad (\text{by } (***))$$

Because g and id_{S^2} are homotopic, $\deg(g) = \deg(id_{S^2})$. By Propositio

13.27, John Lee, "Introduction to Topological manifolds", $\deg(id_{S^2}) =$

Thus, $\deg(g) = 1 \neq 0$. By the same reference, page 368, Propositio

13.28, g is surjective. Thus $(h \circ g)(\eta_s(t))$ is surjective from

$(s,t) \in ([0,1) \times [0,1] / \sim) \simeq S^2$ to S^2. Thus, $r^n \gamma$ is surjective from

$([0,1) \times [0,1) / \sim) \simeq S^2$ to M.

Take any $q \in M \setminus \{p\}$. Because $r^n \gamma$ is surjective from $[0,1]\times[0,1]/\sim$ to M, there exist $s_{n,q}$ and $t_{n,q}$ in $[0,1]$ such that $r^n\gamma(s_{n,q}, t_{n,q}) = q$. Denote $\lambda_n(t) = (r^n\gamma)(s_{n,q}, t) = r^n(\gamma_{s_{n,q}}(t))$. Then $q \in \lambda_n$ for all $n \in \mathbb{N}$.

Also, $\lambda_n|_{[\tau_i, \tau_{i+1})}$ is the shortest geodesic from $\lambda_n(\tau_i)$ to $\lambda_n(\tau_{i+1})$.

The ~~tup~~ tuple $(\lambda_n(\tau_0), \dots, \lambda_n(\tau_{m+1})) \in M^{m+1}$, which is a compact set. Thus, there is a convergent subsequence, namely

$$(\lambda_{n_k}(\tau_0), \dots, \lambda_{n_k}(\tau_{m+1})) \longrightarrow (q_0, \dots, q_{m+1}).$$

We have:

$$\lambda_{n_k}|_{[\tau_i, \tau_{i+1}]} \text{ is the shortest geodesic from } \lambda_{n_k}(\tau_i) \text{ to } \lambda_{n_k}(\tau_{i+1}),$$

$$\lambda_{n_k}(\tau_i) \to q_i, \quad \lambda_{n_k}(\tau_{i+1}) \longrightarrow q_{i+1}.$$

Then by Lemma 1, $\lambda_{n_k}|_{[\tau_i, \tau_{i+1})}$ converges uniformly to a geodesic $\lambda_0|_{[\tau_i, \tau_{i+1}]}$ from q_i to q_{i+1}. Therefore, we get a curve λ_0 on M that starts and ends at p. Because $q \in \lambda_{n_k}$ for all $k \in \mathbb{N}$, $q \in \lambda_0$ as well. Since $p, q \in \lambda_0$ and $p \neq q$, $L(\lambda_0) > 0$. We have

$$L(\lambda_0) = \sum_{i=0}^{m} L(\lambda_0|_{[\tau_i, \tau_{i+1}]}) = \lim_{k\to\infty} \sum_{i=0}^{m} L(\lambda_{n_k}|_{[\tau_i, \tau_{i+1}]})$$

$$= \lim_{k\to\infty} L(\lambda_{n_k}) = \lim_{k\to\infty} L(r^{n_k}(\gamma_{s_{n_k,q}}))$$

We proved in $(**)$ that $L(r^n(\gamma_s)) \to 0$ uniformly in $s \in [0,1]$. Thus, $\lim_{k\to\infty} L(r^{n_k}(\gamma_{s_{n_k,q}})) = 0$. Thus $L(\lambda_0) = 0$, which is a contradiction.