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Math 8386: Calculus of Variations

Homework #1

(1) Lemma 2.2.1, page 133, Jost-Li Jost.

Let $(V, \|.\|)$ be a normed vector space over $\mathbb{R}$ and $f: V \to \mathbb{R}$ be a linear map. Put $\|f\|_* = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}$. We'll show that f is continuous iff and only if $\|f\|_* < \infty$.

($\Rightarrow$) Suppose that f is continuous. Then there exists $\delta > 0$ such that $|f(x)| < 1$ for all $x \in V$, $\|x\| < \delta$. Thus, for all $x \in V$ such that $\|x\| = \frac{\delta}{2}$ we have

$$|f(x)| < 1 = \frac{2}{\delta}\|x\|.$$

Take $y \in V \setminus \{0\}$ arbitrarily. Put $x = \frac{\delta}{2\|y\|} y \in V$. Then $\|x\| = \frac{\delta}{2}$. Thus,

$$|f(x)| < \frac{2}{\delta}\|x\|.$$

Because f is linear, we get $\left|\frac{\delta}{2\|y\|} f(y)\right| = |f(x)| < \frac{2}{\delta}\|x\| = \frac{2}{\delta} \frac{\delta}{2\|y\|}\|y\| = 1$.

Thus, $\frac{\delta}{2\|y\|}|f(y)| < 1$, or equivalently, $|f(y)| < \frac{2}{\delta}\|y\|$. Note that this is true for all $y \in V \setminus \{0\}$. Thus, $\|f\|_* = \sup_{y \neq 0} \frac{|f(y)|}{\|y\|} \leq \frac{2}{\delta} < \infty$.

($\Leftarrow$) Suppose that $C = \|f\|_* < \infty$. Then $|f(x)| \leq C\|x\|$ for all $x \in V$. Let (x_n) be a sequence in V which converges to $x_0 \in V$. We have

$$|f(x_n) - f(x_0)| = |f(x_n - x_0)| \leq C\|x_n - x_0\| \longrightarrow 0 \text{ as } n \to \infty.$$

Therefore, f is continuous in V.

(2) Let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$. Put

$V^* = \{f : V \to \mathbb{R}$ linear and continuous$\}$, which is obviously a vector space.

By the previous problem, for every $f \in V^*$, $\|f\|_* = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} < \infty$.

We'll show that $\|\cdot\|_*$ is a norm on V^*.

If $f \neq 0$ then there exists $x_0 \in V \setminus \{0\}$ such that $f(x_0) \neq 0$. Then

$$\|f\|_* \geq \frac{|f(x_0)|}{\|x_0\|} > 0.$$

If $\lambda \in \mathbb{R}$ and $f \in V^*$ then $\|\lambda f\|_* = \sup_{x \neq 0} \frac{|\lambda f(x)|}{\|x\|} = \sup_{x \neq 0} |\lambda| \frac{|f(x)|}{\|x\|} = |\lambda| \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}$

$$= |\lambda| \, \|f\|_*.$$

Take $f, g \in V^*$ arbitrarily. We want to show that $\|f+g\|_* \leq \|f\|_* + \|g\|_*$. For every $x \in V \setminus \{0\}$,

$$\frac{|(f+g)(x)|}{\|x\|} = \frac{|f(x) + g(x)|}{\|x\|} \leq \frac{|f(x)|}{\|x\|} + \frac{|g(x)|}{\|x\|} \leq \|f\|_* + \|g\|_*.$$

Thus,

$$\|f+g\|_* = \sup_{x \neq 0} \frac{|(f+g)(x)|}{\|x\|} \leq \|f\|_* + \|g\|_*.$$

Therefore, $(V^*, \|\cdot\|_*)$ is a normed vector space.

(3) Let $a, b \geq 0$ and $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We'll show that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \qquad (*)$$

If $a = 0$ or $b = 0$ then $(*)$ is true because the left hand side is zero.

Now we assume $a, b > 0$. $(*)$ is equivalent to

$$\log(ab) \leq \log\left(\frac{a^p}{p} + \frac{b^q}{q}\right),$$

which is equivalent to $\log(a) + \log(b) \leq \log\left(\frac{a^p}{p} + \frac{b^q}{q}\right)$.

This is equivalent to $\frac{1}{p}\log(a^p)+\frac{1}{q}\log(b^q) \leq \log\left(\frac{a^p}{p}+\frac{b^q}{q}\right)$.

This inequality can be proved if we show that the map $\log:(0,\infty)\to\mathbb{R}$ is concave. We have $(\log)'(t)=\frac{1}{t}$, $(\log)''(t)=-\frac{1}{t^2}<0$ for all $t\in(0,\infty)$.

Thus, $\log$ is concave in $(0,\infty)$.

④ ~~Let~~ Problem 2.1, p. 157, Jost-Li Jost.

Let $(V,\|\cdot\|_V)$ and $(W,\|\cdot\|_W)$ be two normed vector spaces. For each linear map $f:V\to W$, we put $\|f\| = \sup_{x\neq 0}\frac{\|f(x)\|_W}{\|x\|_V}$.

(i) Let $f:V\to W$ be a linear map. We'll show that f is continuous if and only if $\|f\|<\infty$.

The proof will be almost the same as in Problem 1.

($\Rightarrow$) Suppose that f is continuous. Then there exists $\delta>0$ such that $\|f(x)\|_W<1$ for all $x\in V$, $\|x\|_V<\delta$. Thus, for all $x\in V$ such that $\|x\|_V=\frac{\delta}{2}$, we have

$$\|f(x)\|_W < 1 = \frac{2}{\delta}\|x\|_V.$$

Take $y\in V\setminus\{0\}$ arbitrarily. Put $x=\frac{\delta}{2\|y\|_V}y \in V$. Then $\|x\|_V=\frac{\delta}{2}$. Thus,

$$\|f(x)\|_W < \frac{2}{\delta}\|x\|_W.$$

Because f is linear, we get

$$\left\|\frac{\delta}{2\|y\|_V}f(y)\right\|_W < \frac{2}{\delta}\|x\|_V = \frac{2}{\delta}\frac{\delta}{2\|y\|_V}\|y\|_V = 1.$$

Thus, $\frac{\delta}{2\|y\|_V}\|f(y)\|_W<1$, or equivalently $\|f(y)\|_W<\frac{2}{\delta}\|y\|_V$. Note that this is true for all $y\in V\setminus\{0\}$. Thus, $\|f\| = \sup_{y\neq 0}\frac{\|f(y)\|_W}{\|y\|_V} \leq \frac{2}{\delta}<\infty$.

($\Leftarrow$) Suppose that $C = \|f\| < \infty$. Then $\|f(x)\|_W \leq C\|x\|_V$ for all $x \in V$.

Let (x_n) be a sequence in V which converges to $x_0 \in V$. We have

$$\|f(x_n) - f(x_0)\|_W = \|f(x_n - x_0)\|_W \leq C\|x_n - x\|_V \to 0 \quad \text{as } n \to \infty.$$

Therefore, f is continuous in V.

(ii) Put $L(V,W) = \{f : V \to W \text{ linear and continuous}\}$. We'll show that $\|.\|$ is a norm on $L(V,W)$.

It is easy to ~~s~~ see that $L(V,W)$ is a vector space (over $\mathbb{R}$). If $f \in L(V,W)$ and $f \neq 0$ then there exists $x_0 \in V \setminus \{0\}$ such that $f(x_0) \neq 0$. Then

$$\|f\| \geq \frac{\|f(x_0)\|_W}{\|x_0\|_V} > 0.$$

If $\lambda \in \mathbb{R}$ and $f \in L(V,W)$ then

$$\|\lambda f\| = \sup_{x \neq 0} \frac{\|\lambda f(x)\|_W}{\|x\|_V} = \sup_{x \neq 0} |\lambda| \frac{\|f(x)\|_W}{\|x\|_V} = |\lambda| \sup_{x \neq 0} \frac{\|f(x)\|_W}{\|x\|_V} = |\lambda| \, \|f\|.$$

Take $f, g \in L(V,W)$ arbitrarily. We want to show that $\|f + g\| \leq \|f\| + \|g\|$.

For every $x \in V \setminus \{0\}$,

$$\frac{\|(f+g)(x)\|_W}{\|x\|_V} = \frac{\|f(x) + g(x)\|_W}{\|x\|_V} \leq \frac{\|f(x)\|_W}{\|x\|_V} + \frac{\|f(x)\|_W}{\|x\|_V} \leq \|f\| + \|g\|.$$

Thus,

$$\|f + g\| = \sup_{x \neq 0} \frac{\|(f+g)(x)\|_W}{\|x\|_V} \leq \|f\| + \|g\|.$$

Therefore, $(L(V,W), \|.\|)$ is a normed vector space.

(iii) Suppose that $(W, \|.\|_W)$ is a Banach space. We'll show that $(L(V,W), \|.\|)$ is also a Banach space.

Let (f_n) be a Cauchy sequence in $(L(V,W), \|\cdot\|)$. We'll show that (f_n) converges.

For every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $\|f_n - f_m\| < \varepsilon$ for all $m,n > N(\varepsilon)$.

Fix some element $x \in V$. We have

$$\|f_n(x) - f_m(x)\|_W = \|(f_n - f_m)(x)\|_W \leq \|f_n - f_m\| \, \|x\|_V \leq \varepsilon \|x\|_V \quad \forall m,n > N(\varepsilon).$$

Thus, the ~~space~~ sequence $(f_n(x))$ is a Cauchy sequence in W. Because W is Banach, $(f_n(x))$ converges in W. We write $f(x) = \lim_{n\to\infty} f_n(x) \in W$. Then we get a well-defined function $f: V \to W$. We have for $x,y \in V$, $c \in \mathbb{R}$,

$$f(x+cy) = \lim_{n\to\infty} f_n(x+cy) = \lim_{n\to\infty} (f_n(x) + cf_n(y)) = \lim_{n\to\infty} f_n(x) + c\lim_{n\to\infty} f_n(y)$$
$$= f(x) + cf(y).$$

Thus, f is linear. Next, we'll show that f is continuous. Since (f_n) is a Cauchy sequence in $(L(V,W), \|\cdot\|)$, it is bounded. Thus, there exists $M \in \mathbb{R}$ such that $\|f_n\| \leq M$ for all $n \in \mathbb{N}$. Thus,

$$\|f_n(x)\|_W \leq M \|x\|_V \quad \forall n \in \mathbb{N}, \forall x \in V.$$

Since $f(x) = \lim_{n\to\infty} f_n(x)$, we have $\|f(x)\|_W = \lim_{n\to\infty} \|f_n(x)\|$. This implies

$$\|f(x)\|_W \leq M \|x\|_V \quad \forall n \in \mathbb{N}, \forall x \in V.$$

Therefore, f is continuous. Now we have $f \in L(V,W)$. Next, we'll show that $\|f_n - f\| \to 0$. For each $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $\|f_n - f_m\| < \varepsilon$ for all $m,n > N(\varepsilon)$. For ~~m,~~ $n > N(\varepsilon)$, ~~we have~~ and $x \in V$, we have

$$\|f_n(x) - f(x)\|_W \leq \|f_n(x) - f_m(x)\|_W + \|f_m(x) - f(x)\|_W$$
$$\leq \underbrace{\|f_n - f_m\|}_{< \varepsilon \text{ if } m > N(\varepsilon)} \cdot \|x\|_W + \|f_m(x) - f(x)\|_W$$

Thus, $\|f_n(x)-f(x)\|_W \leq \varepsilon\|x\|_V + \|f_m(x)-f(x)\|_W \quad \forall m > N(\varepsilon)$.

Let $m\to\infty$. Then $\|f_n(x)-f(x)\|_W \leq \varepsilon\|x\|_V$. Thus,

$$\|f_n - f\| = \sup_{x\neq 0} \frac{\|f_n(x)-f(x)\|_W}{\|x\|_V} \leq \varepsilon \quad \forall n > N(\varepsilon)$$

Therefore, $\lim_{n\to\infty} \|f_n - f\| = 0$.

⑤ Problem 2.2, page 157, Jost-Li-Jost.

Let $(V, \|.\|)$ be a normed linear space with the following property.

$\forall (x_n),(y_n)$ in V such that
- $\limsup_n \|x_n\| \leq 1$, $\limsup \|y_n\| \leq 1$,
- $\lim_{n\to\infty} \|x_n+y_n\| = 2$,

we have $\lim_{n\to\infty}(x_n - y_n) = 0$.

We'll show that $(V,\|.\|)$ is uniformly convex.

Take any $\varepsilon>0$. We show that $\exists\, \delta\in(0,1)$ such that if $x,y\in V$, $\|x\|=\|y\|=1$ and $\|x+y\|>2(1-\delta)$ then $\|x-y\|<\varepsilon$. Suppose otherwise. Then for each $n\in\mathbb{N}, n>1$, there exist $x_n, y_n\in V$ satisfying $\|x_n\|=\|y_n\|=1$ and $\|x_n+y_n\|>2(1-\frac{1}{n})$ and $\|x_n-y_n\|\geq\varepsilon$. It is clear that $\limsup_{n\to\infty}\|x_n\| = \limsup_{n\to\infty}\|y_n\|=1$.

$$2\left(1-\frac{1}{n}\right) < \|x_n+y_n\| \leq \|x_n\|+\|y_n\| = 2 \quad \forall n>1$$

Thus, $\lim_{n\to\infty}\|x_n+y_n\| = 2$. Thus by the hypothesis, we have $\lim_{n\to\infty}(x_n-y_n)=0$.

In particular, $\lim_{n\to\infty}\|x_n-y_n\|=0$. This is a contradiction because $\|x_n-y_n\|\geq\varepsilon$ for every $n\in\mathbb{N}, n>1$.

⑥ Problem 2.3, page 157, Jost–Li Jost.

Let $(V, \|\cdot\|)$ be a uniformly convex normed space. We'll show that $(V, \|\cdot\|)$ is strictly normed. Take $a, b \in V \setminus \{0\}$ such that $\|a+b\| = \|a\| + \|b\|$. We'll show that $a = \alpha b$ for some $\alpha > 0$.

Without loss of generality, we can assume $\|b\| \geq \|a\|$. Put $a_1 = \frac{a}{\|a\|}$ and $b_1 = \frac{b}{\|b\|}$. We have

$$\|a_1 + b_1\| = \left\| \frac{a}{\|a\|} + \frac{b}{\|b\|} \right\| = \left\| \frac{a+b}{\|a\|} - \frac{\|b\| - \|a\|}{\|a\|\|b\|} b \right\|$$

$$\geq \left\| \frac{a+b}{\|a\|} \right\| - \left\| \frac{\|b\| - \|a\|}{\|a\|\|b\|} b \right\|$$

$$= \frac{\|a+b\|}{\|a\|} - \frac{\|b\| - \|a\|}{\|a\|} = 2$$

We have $2 \leq \|a_1 + b_1\| \leq \|a_1\| + \|b_1\| = 1 + 1 = 2$. Thus, $\|a_1 + b_1\| = 2$.

Because V is uniformly convex, for each $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0,1)$ such that if $\|x\| = \|y\| = 1$ and $\|x+y\| > 2(1-\delta_\varepsilon)$ then $\|x-y\| < \varepsilon$. We have $\|a_1\| = \|b_1\| = 1$ and $\|a_1 + b_1\| = 2 > 2(1-\delta_\varepsilon)$. Thus $\|a_1 - b_1\| < \varepsilon$ for any $\varepsilon > 0$.

Thus $a_1 = b_1$. Hence, $\frac{a}{\|a\|} = \frac{b}{\|b\|}$. Therefore, $a = \alpha b$ with $\alpha = \frac{\|a\|}{\|b\|} > 0$.